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Existence of an attractor for a geometric tetrahedron transformation

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We analyze the dynamical properties of a tetrahedron transformation on the space of non-degenerate tetrahedra which can be identified with the non-compact globally symmetric 8-dimensional space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$. We establish the existence of a local attractor which coincides with the set of regular tetrahedra and identify conditions which imply that the basin of attraction is the entire space. In numerical tests, these conditions are fulfilled for a large set of random tetrahedra.

1. Introduction

The research on geometric transformations for tetrahedra has its offspring in the importance of tetrahedral meshes in the computational engineering: physical properties of objects are virtually tested by simulations which discretize the object at the beginning, using mostly a tetrahedral or hexahedral mesh – depending on the purpose. The convergence rate and error estimates of the numerical solution schemes depend severely on the mesh quality (see [1,2]): distorted mesh elements can decelerate or even impede the convergence of

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the numerical method and negatively affect the solution accuracy (see numerical examples in [3]). These are the main reasons why at the beginning of every simulation, a lot of effort is put into the preprocessing in order to guarantee a high quality mesh. It should be remarked that tools for mesh smoothing exist in abundance using a wide range of mathematical methods (see overview in [4]), but at the moment, they do not usually replace in totality the expensive correction of single elements by hand.

Looking for a simple, but effective smoothing method, [5] introduced the geometric element mesh smoothing based on a generic geometric element transformation which relocates the vertices of every mesh element such that it becomes the most regular possible. Its good convergence properties and applicability were proven in a serial of articles (see [6] and references within), also analytically for a single element in [7]. A very comprehensive presentation of this method is given in [8, Section 6.3] comparing its runtime, convergence and smoothing results to the dominating techniques in mesh smoothing.

In this article, we study a different geometric transformation for tetrahedra which is derived from the rotational symmetry group of a regular tetrahedron. It can be the base of a geometric mesh smoothing algorithm analogous to the one mentioned above. In its motivational background, it is strongly related to the geometric triangle transformation introduced in [9]. We call a transformation *geometric* if it commutes with the similarity transformations of the euclidean space, these are translations, homotheties and rotations. Intuitively, the geometric transformations map similar tetrahedra to similar tetrahedra.

Our approach is a dynamical analysis of the tetrahedron transformation Θ: where does a discrete orbit $\left\{\Theta^n(x) \;\middle|\; n \geq 0\right\}$ for an arbitrary tetrahedron x converge to? The analysis is based on two facts: firstly, we define the set of tetrahedra as the special linear group $\mathrm{SL}(3,\mathbb{R})$ which is a non-compact Lie group; secondly, the transformation – being geometric – commutes with the rotational group $\mathrm{SO}(3,\mathbb{R})$. Consequently, we can employ techniques from the theory of compact group actions and the geometric structure of Lie groups and their Lie algebras together with results of the theory of dynamical systems for the analysis of the dynamical behavior of the transformation.

For the understanding of dynamics, attractors play a central role: let T be a continuous map on a topological space X, we then call a compact set $\mathcal{A} \subset X$ which is T-invariant, $T(\mathcal{A}) \subset \mathcal{A}$, an *attractor* for T if there exists an open T-invariant neighborhood $U \supset \mathcal{A}$ such that $\bigcap_{n\geq 0} T^n(U) = \mathcal{A}$. The maximal set U with the property above is called *basin of attraction* because every $x \in U$ eventually converges under positive iterates of T to $\mathcal{A}$. If $U = X$, the attractor is called a *global attractor*. An attractor can have a complicated topological structure, e.g. being a fractal set, and carry a non-trivial dynamics $T|_{\mathcal{A}}$.

The article is organized as following: we define the space of tetrahedra and discuss briefly its properties. In the subsequent section we motivate and introduce the tetrahedron transformation, the main subject of this article, and prove its basic properties. We are then prepared to show the existence of a local attractor and the conditions under which its basin of attraction is the whole space of tetrahedra.

While we focus in this article on a specific transformation, our methods are equally well applicable to study the dynamics of any $\mathrm{SO}(3,\mathbb{R})$-equivariant transformation on $\mathrm{SL}(3,\mathbb{R})$.

2. Space of tetrahedra

Every non-degenerate tetrahedron x with vertices $x_0, x_1, x_2, x_3 \in \mathbb{R}^3$ can be written up to translation as a $(3,3)$-matrix $x = (x_1 - x_0, x_2 - x_0, x_3 - x_0)$ with $\det(x) \neq 0$. Therefore, we consider the set $\mathrm{GL}(3,\mathbb{R})$ of invertible real matrices which is a 9-dimensional open affine subvariety in $\mathbb{R}^9$ with two connected components corresponding to matrices with negative and positive determinant, respectively. Tetrahedra which have the same shape, but different volumes, should not be distinguished. Consequently, as the volume of a tetrahedron x can be computed by $\frac{1}{6}\det(x)$ it suffices to define the set of tetrahedra X by

$$X := \left\{x \in \mathrm{GL}(3,\mathbb{R}) \;\middle|\; \det(x) = 1\right\} = \mathrm{SL}(3,\mathbb{R})$$

which is the *special linear group* and a 8-dimensional submanifold in $\mathbb{R}^9$.