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# Topologizing Lie algebra cohomology

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ABSTRACT

We show that the theory of Lie algebra cohomology can be recast in a topological setting and that classical results, such as the Shapiro lemma and the van Est isomorphism, carry over to this augmented context.



## 1. Introduction

The theory of cohomology for groups and Lie algebras dates back to the pioneering works of, among others, Cartan, Chevalley, Eilenberg, Kozul and Mac Lane [5,6,10], and is by now an indispensable tool in a variety of different branches of mathematics. In recent years, there has been an increasing interest in the topological aspects of group cohomology, since it turns out that there are many instances where one does not have vanishing of the group cohomology on the nose, but only of the *reduced* cohomology (see e.g. [3,17,19] for examples of this phenomenon). When the group $G$ in question is a connected Lie group and the coefficient module is smooth, the van Est theorem provides an (a priori algebraic) isomorphism between the cohomology of $G$ and the (relative) Lie algebra cohomology of its Lie algebra, and keeping in mind the abundance of results involving *reduced* group cohomology, it is natural to ask if also the Lie algebra cohomology carries a canonical topology and, if so, whether or not the van Est isomorphism is actually a homeomorphism. Both questions are answered affirmatively in Section 2 and Section 3, respectively. Along the way, we provide a reasonably self contained introduction to the theory of Lie algebra cohomology, with the hope of making our results more accessible to non-experts. It should also be mentioned that the topological van Est theorem was already stated, and used, in [11] and is an important tool in our ongoing

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project concerning polynomial cohomology of nilpotent Lie groups. Throughout the paper, emphasis will be put on the arguments pertaining to Lie algebra cohomology and even though cohomology of Lie groups is a central theme in Section 3, we will assume familiarity with this theory (however, references will be given whenever appropriate) which can be found in [8, Chapter III] or [4, Chapter X]. We will develop the cohomology theory for Lie algebras within the framework of relative homological algebra, primarily following [8], and in many cases the passage from the algebraic context to the topological one merely consists of making sure that all maps involved respect the topologies.

## 2. Topological Lie algebra cohomology

Throughout this section, $\mathfrak{g}$ denotes a (finite dimensional) Lie algebra over the reals and $\mathfrak{h}$ denotes a Lie sub-algebra of $\mathfrak{g}$. For now, there are no restrictions on $\mathfrak{h}$, but in order to develop the cohomology theory of $\mathfrak{g}$ relative to $\mathfrak{h}$ we will soon require $\mathfrak{h}$ to be reductive; see section 2.2 and Remark 2.8.

**Definition 2.1.** A continuous (or topological) $\mathfrak{g}$-module is a Hausdorff topological vector space (t.v.s.) $E$ with an action of the Lie algebra $\mathfrak{g}$ such that each $X \in \mathfrak{g}$ acts as a continuous operator. A morphism of continuous $\mathfrak{g}$-modules (also simply referred to as a $\mathfrak{g}$-morphism) is a continuous, linear map of t.v.s. which intertwines the $\mathfrak{g}$-actions. A vector $\xi \in E$ is said to be $\mathfrak{g}$-invariant if $X.\xi = 0$ for all $X \in \mathfrak{g}$ and the set of $\mathfrak{g}$-invariant vectors is denoted $E^{\mathfrak{g}}$.

By the universal property of the enveloping algebra $\mathcal{U}(\mathfrak{g})$, any Lie algebra representation extends to an algebra representation of $\mathcal{U}(\mathfrak{g})$, and if the representation of $\mathfrak{g}$ is by continuous operators on a t.v.s., then so is the induced representation of $\mathcal{U}(\mathfrak{g})$. In what follows, we will freely identify the representation of $\mathfrak{g}$ with the corresponding representation of $\mathcal{U}(\mathfrak{g})$. The key to getting homological algebra working in this topological context is to pin-point the right definition of morphisms and injective modules, which we will adapt, mutatis mutandis, from the corresponding theory for groups [8, Chapter III]:

**Definition 2.2.** Let $E$ and $F$ be continuous $\mathfrak{g}$-modules. An injective $\mathfrak{g}$-morphism $f\colon E \to F$ is said to be $\mathfrak{h}$-*strengthened* if there exists a continuous $\mathfrak{h}$-equivariant map $s\colon F \to E$ such that $s \circ f = \mathrm{id}_E$. A general $\mathfrak{g}$-morphism $f\colon E \to F$ is said to be $\mathfrak{h}$-strengthened if both the inclusion $\ker(f) \to E$ and the induced map $E/\ker(f) \to F$ are $\mathfrak{h}$-strengthened in the sense just defined. Lastly, a $\mathfrak{g}$-morphism $f\colon E \to F$ is called *strengthened* if is $\mathfrak{h}$-strengthened with respect to the trivial subalgebra.

**Definition 2.3.** A continuous $\mathfrak{g}$-module $E$ is called $\mathfrak{h}$-*relative injective* if for any two other such modules $A, B$, any $\mathfrak{h}$-strengthened injective $\mathfrak{g}$-morphism $\iota\colon A \to B$ and any $\mathfrak{g}$-morphism $f\colon A \to E$ there exists a $\mathfrak{g}$-morphism $\tilde{f}\colon B \to E$ such that $f = \tilde{f} \circ \iota$. When the subalgebra $\mathfrak{h}$ is the trivial one, we simply refer to $\mathfrak{h}$-relative injective modules as being *relative injective*.

One may now consider $\mathfrak{h}$-strengthened, $\mathfrak{h}$-relative injective resolutions of a given $\mathfrak{g}$-module $E$ and an adaptation of the standard arguments from homological algebra (carried out in detail in Appendix A), implies that given any two such resolutions, upon passing to $\mathfrak{g}$-invariants and thereafter to cohomology, the resulting cohomology spaces are isomorphic in each degree—the isomorphism being as (generally non-Hausdorff!) topological vector spaces. Thus, if we can show that any continuous $\mathfrak{g}$-module $E$ admits such a resolution, then the cohomology of $\mathfrak{g}$, relative to $\mathfrak{h}$, with coefficients in $E$ is well defined as a topological object; in what follows we provide such a resolution under the mild additional assumption that $E$ is locally convex.