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Non-preserved curvature conditions under constrained mean curvature flows

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We provide explicit examples which show that mean convexity (i.e. positivity of the mean curvature) and positivity of the scalar curvature are non-preserved curvature conditions for hypersurfaces of the Euclidean space evolving under either the volume- or the area preserving mean curvature flow. The relevance of our examples is that they disprove some statements of the previous literature, overshadow a widespread *folklore conjecture* about the behaviour of these flows and bring out the discouraging news that a traditional singularity analysis is not possible for constrained versions of the mean curvature flow.

1. Introduction and main results

We consider a closed n-dimensional manifold M smoothly embedded as a hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ by means of the parametrization $F_0 : M \to \mathbb{R}^{n+1}$. We deform $M_0 := F_0(M)$ according to a *constrained Mean Curvature Flow* (C-MCF), that is, we consider a family of isometric immersions $F : M \times [0,T) \to \mathbb{R}^{n+1}$ solving the initial value problem:

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) = \big(h(t) - H(p,t)\big)\, N(p,t), \quad p \in M,\ t \in (0,T) \\ F(\cdot,0) = F_0 \end{cases}, \tag{1.1}$$

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where N is the unit normal vector field pointing outward (if each evolving hypersurface $M_t = F(M, t)$ encloses a domain Ω_t) and H the mean curvature of M_t. More precisely, hereafter we study two types of constrained flows depending on the choice of the global term $h(t)$:

- The *Volume Preserving Mean Curvature Flow* (VP-MCF) is obtained by defining $h(t)$ as the average mean curvature:

$$h(t) = \overline{H} = \frac{\int_M H d\mu_t}{|M_t|}, \tag{1.2}$$

  where $d\mu_t$ is the induced Riemannian volume element of M_t, and $|M_t| = \int_M d\mu_t$ denotes the corresponding n-volume (which we shall call *area*).
- The *Area Preserving Mean Curvature Flow* (AP-MCF) is defined by choosing

$$h(t) = \frac{\int_M H^2 \, d\mu_t}{\int_M H \, d\mu_t}. \tag{1.3}$$

The presence of the global term $h(t)$ in the flow equation (1.1) has two major consequences:

(1) for $h(t)$ defined as in (1.2) the flow keeps the enclosed volume constant while the area decreases as time evolves. On the other hand, if we choose $h(t)$ as in (1.3), any closed hypersurface moves preserving its area and increasing the volume enclosed by the hypersurface;
(2) makes the usual techniques in geometric flows (e.g. the application of maximum principles) either fail or become more subtle.

The resultant evolution problem (see Section 2 for a brief account of the previous literature) is particularly appealing—since from (1) it is specially well suited for applications to the isoperimetric problem—and challenging because (2) causes numerous extra complications; for instance, the comparison principle, a basic property for the ordinary mean curvature flow (MCF) fails in general for (1.1), e.g., an initially embedded curve may develop self-intersections (cf. [18]) under VP-MCF. Hence the present knowledge of (1.1) is considerably poorer than that of the unconstrained evolution. This is even more evident when we talk about the analysis of singularities.

For the unconstrained MCF a systematic study was started by G. Huisken in [12]; in the latter and subsequent papers (e.g. [13]) the preservation of the mean convexity ($H > 0$) played a key role. So a natural first step towards a similar analysis of singularities for the non-local flows would be to wonder if mean convexity is preserved or not under (1.1). There is (even written) evidence that many experts were inclined towards believing in a positive answer:

(a) The preservation of $H > 0$ under VP-MCF was claimed and *proved* in [22, Lemma 3.11].
(b) The corresponding statement and *proof* for the AP-MCF is indeed published in [21, Lemma 10].
(c) It is a folklore conjecture that *if the average mean curvature is bounded, we expect that the behaviour of singularities for the* VP-MCF *after parabolic rescaling is the same as by* MCF (quote adapted from [3]).

Roughly speaking, in this note we give counterexamples to the *statements* in (a) and (b), that is, we find mean convex surfaces in $\mathbb{R}^3$ which evolve under AP- or VP-MCF to surfaces with negative mean curvature at some points. Let us remark that, despite (a) and (b) study (1.1) in a certain non-Euclidean ambient space, the ambient curvature plays no role in their *proof* of preservation of $H > 0$.

Furthermore, our examples of loss of mean convexity are all rotationally symmetric closed surfaces for which the average mean curvature is known to be bounded (cf. [4, Lemma 8.4]). Accordingly our examples