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Geometric properties of homogeneous parabolic geometries with generalized symmetries

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ABSTRACT

We investigate geometric properties of homogeneous parabolic geometries with generalized symmetries. We show that they can be reduced to a simpler geometric structures and interpret them explicitly. For specific types of parabolic geometries, we prove that the reductions correspond to known generalizations of symmetric spaces. In addition, we illustrate our results on an explicit example and provide a complete classification of possible non-trivial cases.

1. Introduction

The reader should be familiar with the theory of parabolic geometries, cf. [1]. We will always consider the parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) on the connected smooth manifold M satisfying the following assumptions: The group G is simple (not necessarily connected) Lie group, the parabolic geometry

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$(\mathcal{G} \to M, \omega)$ is regular and normal, and its automorphism group $\mathrm{Aut}(\mathcal{G}, \omega)$ acts transitively on M, i.e., the parabolic geometry is homogeneous.

Let us fix arbitrary $x_0 \in M$ for the rest of the article and denote by $\mathrm{Aut}(\mathcal{G}, \omega)_{x_0}$ the stabilizer of x_0 in $\mathrm{Aut}(\mathcal{G}, \omega)$. Then any element of $\mathrm{Aut}(\mathcal{G}, \omega)_{u_0}$ can be identified with $g_0 \exp(Z)$ for g_0 in G_0 and $\exp(Z)$ in the unipotent radical of P, where G_0 is Levi part of a chosen reductive Levi decomposition of P. Let us point out that reductive Levi decomposition of P always exists, see [1, Theorem 3.1.3], and we later fix one by choice of grading of $\mathfrak{g}$. Nevertheless, since all reductive Levi subgroups are conjugated by elements of P, the following definition does not depend on the choice of the Levi subgroup $G_0 \subset P$.

Definition 1. Let s be an element of the center $Z(G_0)$ of the Levi subgroup $G_0 \subset P$. We say that the automorphism $\phi \in \mathrm{Aut}(\mathcal{G}, \omega)_{x_0}$ is *s-symmetry* at x_0 if there is $u_0 \in \mathcal{G}$ covering x_0 such that $\phi(u_0) = u_0 s$. All s-symmetries at x_0 for all possible elements s in $Z(G_0)$ together are called *generalized symmetries* at x_0.

We gave in [7, Theorem 4.1.] a significant condition for the existence of generalized symmetries on homogeneous parabolic geometries. To formulate this condition here, we need to introduce important choices and notation:

We fix the restricted root system of $\mathfrak{g}$ in which $\mathfrak{p}$ is a standard parabolic subalgebra of $\mathfrak{g}$, and denote by α_i the positive simple restricted roots numbered according to the convention from [11,1]. We denote by Ξ the set of simple restricted roots corresponding to $\mathfrak{p}$ and we use the notation $\mathfrak{p}_\Xi := \mathfrak{p}$, because we will work with several different parabolic subalgebras of $\mathfrak{g}$ later and we will need to distinguish between them. We denote by $\mathfrak{g}_{\Xi,i}$ the corresponding $|k|$-grading of $\mathfrak{g}$ by Ξ-heights, i.e., $\mathfrak{g}_{\Xi,0}$ is Lie algebra of G_0, and we use the notation $\mathfrak{g}_{\Xi,-}$ and $\mathfrak{p}_{\Xi,+}$ for the negative and positive parts of the grading.

- We denote by $\mathfrak{g}_\gamma$ the root space of the root γ, and we denote by $V_{\Xi,\gamma}$ the indecomposable G_0-submodule of $\mathfrak{g}$ containing the root space $\mathfrak{g}_\gamma$.

Let us point out that the representation Ad of G_0 on $\mathfrak{g}$ is completely reducible and both $\mathfrak{g}_{\Xi,-}$ and $\mathfrak{p}_{\Xi,+}$ decompose into the sums of the modules $V_{\Xi,\gamma}$, but there can be indecomposable G_0-submodules of $\mathfrak{g}_{\Xi,0}$ that are not of the form $V_{\Xi,\gamma}$ for some restricted root γ.

Since these decompositions and their description play a crucial role in the text, let us demonstrate them in several examples on a particular type of parabolic geometries – the so-called generalized path geometries, see [1, Section 4.4.3].

Example 1. Consider $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ for $n \geq 2$ and choose $\Xi = \{\alpha_1, \alpha_2\}$. Then $\mathfrak{g}_{\Xi,-} = V_{\Xi,-\alpha_1-\alpha_2} \oplus V_{\Xi,-\alpha_1} \oplus V_{\Xi,-\alpha_2}$ and $\mathfrak{p}_{\Xi,+} = V_{\Xi,\alpha_1+\alpha_2} \oplus V_{\Xi,\alpha_1} \oplus V_{\Xi,\alpha_2}$. Moreover, if $n > 2$, then $\mathfrak{g}_{\Xi,0} = V_{\Xi,\alpha_3} \oplus \mathbb{R} = V_{\Xi,-\alpha_3} \oplus \mathbb{R}$, and if $n = 2$, then $\mathfrak{g}_{\Xi,0} = \mathbb{R} \oplus \mathbb{R}$ is the Cartan subalgebra.

Moreover, the map Ad_s for $s \in Z(G_0)$ is a certain multiple of identity on each submodule $V_{\Xi,\gamma}$. Let us remark that we do not assume that $\mathrm{Ad} : Z(G_0) \to Gl(\mathfrak{g})$ is injective contrary to the article [7], in which the injectivity also poses no restriction and only makes the article [7] less technical. Indeed, since $\omega = \mathrm{Ad}_{g_0}^{-1} \circ \omega = (r^{g_0})^* \omega$ holds for $g_0 \in Ker(\mathrm{Ad}_{Z(G_0)})$ and thus $Ker(\mathrm{Ad}_{Z(G_0)})$ is subgroup of $\mathrm{Aut}(\mathcal{G}, \omega)$ consisting of automorphisms of $(\mathcal{G} \to M, \omega)$ covering identity on M, the results for $Ker(\mathrm{Ad}_{Z(G_0)}) \backslash \mathrm{Aut}(\mathcal{G}, \omega)$ from [7] can be extended to our situation and we get the following result from [7, Theorem 4.1.]. Let us point out that the assumption $\mathfrak{g}$ simple implies that the group of automorphisms covering identity on M, which we call trivial automorphisms, is countable and discrete.

Proposition 1.1. *There is s-symmetry $\phi \in \mathrm{Aut}(\mathcal{G}, \omega)$ at x_0 such that $\mathrm{Ad}_s|_{V_{\Xi,\alpha_i}} = j_i \cdot \mathrm{id}_{V_{\Xi,\alpha_i}}$ for each $\alpha_i \in \Xi$ if and only if there is automorphism $\phi' \in \mathrm{Aut}(\mathcal{G}, \omega)_{x_0}$ such that $T_{x_0}\phi'$ acts as $j_i^{-1} \cdot \mathrm{id}$ on the distinguished*