Differential Geometry and its Applications 49 (2016) 473–495

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Constant mean curvature hypersurfaces with constant angle in semi-Riemannian space forms

Matias Navarro [a], Gabriel Ruiz-Hernández [b], Didier A. Solis [a,*]

[a] *Facultad de Matemáticas, Universidad Autónoma de Yucatán, Periférico Norte, Tablaje 13615, C.P. 97110, Mérida, Mexico*

[b] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Campus Juriquilla, C.P. 76230, Querétaro, Mexico*

A R T I C L E I N F O

Article history:
Received 4 July 2016
Available online xxxx
Communicated by E. Garcia Rio

MSC:
53B25
53B30
53C42
53C50

Keywords:
Constant angle hypersurfaces
Constant mean curvature
Closed and conformal vector field

A B S T R A C T

We study constant angle semi-Riemannian hypersurfaces M immersed in semi-Riemannian space forms, where the constant angle is defined in terms of a closed and conformal vector field Z in the ambient space form. We show that such hypersurfaces belong to the class of hypersurfaces with a canonical principal direction. This property is a type of rigidity. We further specialize to the case of constant mean curvature (CMC) hypersurfaces and characterize them in two relevant cases: when the hypersurface is orthogonal to Z then it is totally umbilical, whereas if Z is tangent to the hypersurface then it has zero Gauss–Kronecker curvature and either its mean curvature vanishes or the ambient is a semi-Euclidean space. We also treat in detail the surface case, giving a full characterization of the constant angle CMC surfaces immersed in all three dimensional space forms. They are isoparametric surfaces with constant principal curvatures when the ambient is flat. If the mean curvature of the surface is not $\pm 2/\sqrt{3}$ they are either totally umbilic or totally geodesic. In particular when the surface has zero mean curvature it is totally geodesic.

1. Introduction

The study of the geometry of constant angle surfaces is as old as the classical theory of surfaces itself. As its name suggests, constant angle surfaces are immersed surfaces $M^2 \subset \mathbb{R}^3$ which make a constant angle with respect to a distinguished vector field Z. These surfaces can be thought as two dimensional analogues of well known curves that make constant angle with a prescribed direction, such as helices (fixed direction), logarithmic spirals (radial direction) or rhumb lines (direction given by meridians on the sphere). In this

* Corresponding author.
E-mail addresses: matias.navarro@correo.uady.mx (M. Navarro), gruiz@matem.unam.mx (G. Ruiz-Hernández), didier.solis@correo.uady.mx (D.A. Solis).

http://dx.doi.org/10.1016/j.difgeo.2016.10.002
0926-2245/

context, one of the most fundamental questions in this field consists in classifying those constant angle surfaces that satisfy certain geometrical properties, such as being minimal, umbilical, etcetera.

In recent years, a renewed interest for such surfaces has grown and several interesting generalizations of the concept have arisen. For instance, the ambient space $\mathbb{R}^3$ has been replaced by cartesian products or warped products, or even Lorentzian 3-manifolds. Just to mention a few of the most relevant developments in this scenario, we have the classification of surfaces in $\mathbb{R} \times \mathbb{S}^2$ and $\mathbb{R} \times \mathbb{H}^2$ making a constant angle with respect to a constant field in the first factor [6,7,9]; or non-degenerate surfaces in Lorentz–Minkowski 3-space $\mathbb{R}^3_1$ making a constant angle with a fixed non-lightlike direction [20,17]. Going one step further, notice that in the aforementioned cases the distinguished directions project to principal directions on M with non-vanishing principal curvatures. Surfaces having this property are said to have a canonical principal direction and have been extensively studied both in the Riemannian and Lorentzian settings [5,8,13].

Another suitable generalization consists in replacing constant (i.e. parallel) directions for other types of distinguished vector fields that carry geometrical relevance, such as Killing fields [22]. Closed and conformal vector fields are a natural generalization of both parallel and Killing vector fields, since they are infinitesimal generators of conformal mappings that are locally gradient fields. The presence of a closed and conformal vector field in the ambient space is a powerful tool that can be used to establish classification results, as is illustrated by the work of S. Montiel [21] and A. Barros et al. [2] on spacelike hypersurfaces of constant mean curvature. In [16] hypersurfaces in Riemannian warped products making a constant angle with respect to a closed and conformal vector field are classified. In [24] and [1] similar classification results are established for Lorentzian 3-manifolds and Riemannian 3-dimensional space-forms, respectively.

In this work we study constant angle CMC hypersurfaces immersed in semi-Riemannian space forms, thus extending some of the results obtained in [1,4,13,11,16,17,20,22,24] to the context of closed and conformal vector fields. This paper is organized as follows. In section 2 we establish the notation and main results pertaining semi-Riemannian space forms and closed and conformal vector fields. In particular, we show that in any semi-Riemannian space form, any tangent vector can be extended locally to a closed and conformal vector field, which is essentially the projection of a parallel vector field. Moreover, in a space form of non-vanishing curvature, any closed and conformal vector field can be realized in such a way. In this section we also find formulae for the (intrinsic) Laplacian of the squared norm of a closed and conformal vector field defined along a semi-Riemannian hypersurface. In section 3 we develop the concept of a constant angle hypersurface in a semi-Riemannian manifold and prove under mild assumptions that hypersurfaces that make a constant angle with respect to a closed and conformal vector field have a canonical principal direction. In section 4 we deal with two special cases of geometric significance, namely, when Z is either orthogonal or tangent to M, and provide a full description in the semi-Euclidean scenario. Finally, in section 5 we present the classification of CMC surfaces in three dimensional semi-Riemannian space forms having a constant angle with respect to a closed and conformal vector field.

2. Preliminaries

Let us denote by $\mathbb{R}^{n+2}_s$ the semi-Euclidean space given by the real vector space $\mathbb{R}^{n+2}$ endowed with the semi-Riemannian metric

$$\langle u, v\rangle = -u_1 v_1 - \ldots - u_s v_s + u_{s+1} v_{s+1} + \ldots + u_{n+2} v_{n+2},$$

and recall that a vector $u \in T_p\mathbb{R}^{n+2}_s$ is called *timelike, spacelike* or *lightlike* if $\langle u, u\rangle$ is negative, positive, or zero, respectively. Furthermore, if u is non-lightlike then $\epsilon_u = \pm 1$ will denote the sign of $\langle u, u\rangle$.

The non-degenerate hyperquadrics in $\mathbb{R}^{n+2}_s$ are totally umbilical and geodesically complete hypersurfaces with constant sectional curvature, and thus are locally isometric to a semi-Riemannian space form although