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On first integrals of the geodesic flow on Heisenberg nilmanifolds

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ABSTRACT

In this paper we study the geodesic flow on nilmanifolds equipped with a left-invariant metric. We write the underlying definitions and find general formulas for the Poisson involution. As an application we develop the Heisenberg Lie group equipped with its canonical metric. We prove that a family of first integrals giving the complete integrability can be read off at the Lie algebra of the isometry group. We also explain the complete integrability for any invariant metric and on compact quotients.

1. Introduction

Given a smooth manifold M, any complete Riemannian structure $\langle\cdot,\cdot\rangle$ induces the geodesic flow $\Gamma\colon M\times\mathbb{R}\to M$ which can be defined as the Hamiltonian flow associated to the energy function $E(v) := 1/2\langle v, v\rangle$ on TM. Usually, this flow is not integrable in the sense of Liouville and it is generally expected that the integrability of the geodesic flow imposes important obstructions to the topology of the supporting manifold. However contrasting some results of Taimanov [19,27,28] on topological obstructions for real-analytic manifolds supporting real-analytic integrable geodesic flows with some smooth

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examples of smoothly integrable geodesic flows on manifolds that do not satisfy the above obstructions constructed by Butler [8], and Bolsinov and Taimanov [7], we observe that the regularity of first integrals plays a fundamental role that nowadays is not completely well understood. For that reason, when dealing with locally homogeneous manifolds to have the possibility of constructing real-analytic first integrals is very desirable.

In the advances reached in the theory of Hamiltonian systems in the 1980's one can recognize the role of Lie theory in the study of several examples. This is the case of the so known Adler–Kostant–Symes [2,18,26] scheme used for the study of some mechanical systems and of the so known Thimm's method for the study of the geodesic flow [29]. In both cases the main examples arise from semisimple Lie groups. These results appeared parallel to the many studies given by the Russian school which can be found for instance in [14]. Examples of integrable geodesic flows and other systems on Lie groups and quotients or bi-quotients can be found in [6,3–5,8,16,17,21,23,24].

For other Hamiltonian systems on non-semisimple Lie groups only few examples and generalizations are known in the case of the geodesic flow. This is the situation for nilpotent and solvable Lie groups or even their compact quotients which are locally homogeneous manifolds. For instance, Eberlein started a study of the geometry concerning the geodesic flow on Lie groups following his own and longer study in this topic, giving a good material and references in [12,13]. This study of Eberlein is much more general and is mixed with many other geometrical questions.

In the case of 2-step nilpotent Lie groups and their compact quotients there are some interesting results. On the one hand Butler [9] proved the Liouville integrability of the geodesic flow whenever the Lie algebra is Heisenberg–Reiter. On the other hand he also proved the non-commutative integrability – in the sense of Nekhoroshev [22] – whenever the Lie algebra is almost non-singular. Since Bolsinov and Jovanovic [4] proved that integrability in the non-commutative sense implies Liouville integrability, the previous results of Butler give the Liouville integrability for an important family of 2-step nilpotent Lie groups and their compact quotients.

In the present paper we concentrate in the geodesic flow of Lie groups endowed with a metric invariant by left-translations. In the first part we write the basic definitions and get general conditions and formulas for the involution of first integrals making use of the Lie theory tools, that is assuming some natural identifications. We put special emphasis on 2-step nilpotent Lie groups, we take as nilmanifold after [31], for which there exists a developed geometrical theory and several examples and applications see [11].

We apply the results we get to the case of the Heisenberg Lie group H_n of dimension $2n+1$ equipped with the canonical left-invariant metric. Although this is a naturally reductive space the methods of Thimm do not apply directly in this case, since the full isometry group is not semi-simple. Compare also with the Mishchenko–Fomenko method [14,25].

Our main goal is to investigate the nature of the first integrals one can construct. We prove that all the first integrals we get can be visualized on the isometry group. In fact this is the case of quadratic polynomials which are invariant, so as first integrals arising from Killing vector fields. Recall that given a Killing vector field X^* on a Riemannian manifold M one has a first integral on the tangent Lie bundle TM defined by $f_{X^*}(v) = \langle X^*, v\rangle$. We proved that

(i) There is a bijection between the set of quadratic first integrals of the geodesic flow on H_n – with the canonical metric – and the Lie subalgebra of skew-symmetric derivations of the Heisenberg Lie algebra $\mathfrak{h}_n$, so that involution of quadratic first integrals would correspond to a torus of skew-symmetric derivations – Theorem 3.2. Actually a general formulation of quadratic polynomials on a 2-step nilpotent Lie algebras to be first integrals is found so as the pairwise commutativity condition.

(ii) The linear morphism $X^* \mapsto f_{X^*}$ builds a Lie algebra isomorphism onto the image. This is the first example we found of this situation.