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# Energy of generalized distributions

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ABSTRACT

We consider the energy of smooth generalized distributions and also of singular foliations on compact Riemannian manifolds for which the set of their singularities consists of a finite number of isolated points and of pairwise disjoint closed submanifolds. We derive a lower bound for the energy of all $q$-dimensional almost regular distributions, for each $q < \dim M$, and find several examples of foliations which minimize the energy functional over certain sets of smooth generalized distributions.



## 1. Introduction

Let $\sigma : x \in M \mapsto \sigma(x) \subset T_xM$ be a smooth *generalized distribution* [12] on an $n$-dimensional compact and connected Riemannian manifold $(M, g)$. Denote by $M_r$ the subset of its regular points. When the restriction $\sigma_r$ of $\sigma$ to $M_r$ is a regular distribution, that is, the lower semicontinuous function $d$, given by $d(x) = \dim \sigma(x)$, is a constant $q$ on $M_r$, $\sigma$ is said to be an $q$-dimensional *almost regular distribution*. If $d$ is constant on whole $M$, $\sigma$ is the classical distribution. For the sake of brevity, we will refer to generalized distributions simply as *distributions* and we will use the term *regular* for the classical distribution.

In the general case, we can only guarantee that $d$ is constant on each one of the connected components of $M_r$. Let $1 \leq q_1 < q_2 < \cdots < q_l \leq n = \dim M$ be the values of $d$ on $M_r$ and put $M_r^i := \{x \in M_r \mid d(x) = q_i\}$, $i \in \{1, \ldots, l\}$, the union of the connected components, supposed to be oriented, on which $d$ is constant equals to $q_i$. Then $\sigma_r$ is the union of the regular $q^i$-dimensional distributions $\sigma^i$ on $M_r^i$, $i = 1, \ldots, l$, and it could be

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seen as a smooth section of the Grassmannian bundle $\pi : G(M_r) = \bigcup_{i=1}^{l} G_{q_i}(M_r^i) \to M_r$, where $G_{q_i}(M_r^i) = \bigcup_{x\in M_r^i} G_{q_i}(T_x M_r^i)$ is the Grassmannian bundle of the $q_i$-dimensional linear subspaces in the tangent space $TM_r^i$. Because $G_q(M_r^i)$ is diffeomorphic to the *homogeneous fiber bundle* $\mathcal{SO}(M_r^i)/S(O(q_i) \times O(n-q_i))$, $\mathcal{SO}(M_r^i)$ being the principal $SO(n)$-bundle of oriented orthonormal frames of $(M_r^i, g_{M_r^i})$, $G(M_r)$ will be provided (see Section 3) with a natural Riemannian metric $g^K$, known as the *Kaluza–Klein metric* [19]. It makes $\pi : (G(M_r), g^K) \to (M_r, g_{M_r})$ a Riemannian submersion with totally geodesic fibers, where $g_{M_r}$ denotes the induced metric by $g$ on $M_r$.

The energy of a $q$-dimensional regular distribution $\sigma$ is defined in [8] (see also [6] and [18]) as the energy of the map $\sigma : (M, g) \to (G_q(M), g^K)$. An equivalent definition for oriented regular distributions, considered as sections of the bundle of unit decomposable $q$-vectors equipped with the *generalized Sasaki metric*, is given in [4] and [7], among others. For an arbitrary map $\sigma : (M, g) \to (N, h)$ between Riemannian manifolds, $M$ being compact and oriented, the *energy* of $\sigma$ is the integral

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \operatorname{trace} L_\sigma \, dv_M, \tag{1.1}$$

where $L_\sigma$ is the $(1,1)$-tensor field determined by $(\sigma^* h)(X, Y) = g(L_\sigma X, Y)$, for all vector fields $X, Y$, and $dv_M$ denotes the volume form on $(M, g)$. (For more information about the energy functional see [14].) In this paper we define the energy functional of smooth distributions whose set of singular points $M_s = M \setminus M_r$ is given by the union

$$M_s = \{x_1, \dots, x_a\} \cup \Big( \bigcup_{\beta=1}^{b} P_\beta \Big) \tag{1.2}$$

of a finite number of points $x_1, \dots, x_a$ of $M$ and of pairwise disjoint topologically embedded submanifolds $P_\beta$, $\beta = 1, \dots, b$, with $1 \le \dim P_\beta \le n-1$. In fact, the main purpose is not the study of variational problems of this functional, but of the energy itself, of smooth distributions and singular foliations, as a natural extension from the theory about the energy of unit vector fields, with singularities as in (1.2), developed initially by Brito and Walczak in [3] and, later, by Boeckx, Vanhecke and the author in [1]. Note that a unit vector field with singularities determines an *oriented* one-dimensional almost regular distribution.

In terms of $G$-structures, each regular distribution $\sigma^i$, $i = 1, \dots, l$, corresponds with a (unique) $S(O(q_i) \times O(n-q_i))$-reduction of $\mathcal{SO}(M_r^i)$. Then, using the *intrinsic torsion* $\xi$ of these $S(O(q_i) \times O(n-q_i))$-structures, the energy $\mathcal{E}(\sigma)$ of $\sigma$ is expressed in Section 3 as

$$\mathcal{E}(\sigma) = \frac{n}{2}\operatorname{Vol}(M, g) + \frac{1}{4}\int_M \|\xi\|^2 dv_M.$$

If $\sigma$ is completely integrable, the connected components of the maximal integral submanifolds are the leaves of a foliation $\mathcal{F} = \mathcal{F}_\sigma$ with singularities known as a *Stefan foliation* or a *singular foliation* [13]. The energy $\mathcal{E}(\mathcal{F})$ of $\mathcal{F}$ is defined as the energy $\mathcal{E}(\sigma)$ of the *tangent distribution* $\sigma$ of $\mathcal{F}$.

In Section 4, a useful integral formula for almost regular distributions, with finite energy and $M_r$ connected, of its *mixed scalar curvature* and of its *second mean curvatures* is obtained. This formula, which can be seen as a generalization the one given in [1], see also [3], for unit vector fields with singularities, plays a central role for the determination of some lower bounds of the energy of smooth distributions. Thus, in Section 5 we show that tori are the unique compact oriented surfaces admitting a one-dimensional almost regular distribution with finite energy, and for $n \ge 3$, we derive a lower bound for the energy in the set of all $q$-dimensional almost regular distributions, for each $q = 1, \dots, n-1$. As an application of this last result, we find in Section 6 some special classes of foliations, as tubular and radial foliations around points or