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Pure spinors, intrinsic torsion and curvature in even dimensions

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A B S T R A C T

We study the geometric properties of a $2m$-dimensional complex manifold $\mathcal{M}$ admitting a holomorphic reduction of the frame bundle to the structure group $P \subset \mathrm{Spin}(2m, \mathbb{C})$, the stabiliser of the line spanned by a pure spinor at a point. Geometrically, $\mathcal{M}$ is endowed with a holomorphic metric g, a holomorphic volume form, a spin structure compatible with g, and a holomorphic pure spinor field ξ up to scale. The defining property of ξ is that it determines an almost null structure, i.e. an m-plane distribution $\mathcal{N}_\xi$ along which g is totally degenerate.

We develop a spinor calculus, by means of which we encode the geometric properties of $\mathcal{N}_\xi$ corresponding to the algebraic properties of the intrinsic torsion of the P-structure. This is the failure of the Levi-Civita connection ∇ of g to be compatible with the P-structure. In a similar way, we examine the algebraic properties of the curvature of ∇.

Applications to spinorial differential equations are given. In particular, we give necessary and sufficient conditions for the almost null structure associated to a pure conformal Killing spinor to be integrable. We also conjecture a Goldberg–Sachs-type theorem on the existence of a certain class of almost null structures when $(\mathcal{M}, g)$ has prescribed curvature.

We discuss applications of this work to the study of real pseudo-Riemannian manifolds.

1. Introduction

Let $\mathcal{M}$ be a complex manifold of dimension n, and denote by $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ its holomorphic tangent and cotangent bundles respectively, and by $\mathrm{F}\mathcal{M}$ its holomorphic frame bundle. Following [28], we define a *holomorphic metric* on $\mathcal{M}$ to be a non-degenerate holomorphic section g of the bundle $\odot^2\mathrm{T}^*\mathcal{M}$ — here $\odot$ denotes the symmetric tensor product. We identify $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ by means of g. The pair $(\mathcal{M}, g)$ will be referred to as a *complex Riemannian manifold*, and is characterised equivalently by a holomorphic reduction of the structure group of $\mathrm{F}\mathcal{M}$ to the complex orthogonal group $\mathrm{O}(n, \mathbb{C})$. Analogously to real pseudo-Riemannian geometry, there is a unique torsion-free holomorphic affine connection ∇ preserving g,

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also referred to as the *Levi-Civita connection* of g, with associated curvature tensors, which depend holomorphically on $\mathcal{M}$. We shall also assume the existence of a global *holomorphic volume form* $\varepsilon \in \Gamma(\wedge^n \mathrm{T}^* \mathcal{M})$ normalised to $g(\varepsilon, \varepsilon) = n!$ — here, we have extended g to a non-degenerate bilinear form on the bundle $\wedge^\bullet \mathrm{T}\mathcal{M}$ of holomorphic differential forms, and its dual. This induces a further holomorphic reduction of the structure group of $\mathrm{F}\mathcal{M}$ to the complex special orthogonal group $\mathrm{SO}(n, \mathbb{C})$. The pair (g, ε) can be used to define a holomorphic Hodge duality operator $\star$ on $\wedge^\bullet \mathrm{T}^*\mathcal{M}$. We shall henceforth assume $n = 2m$. Then $\star$ squares to plus or minus the identity on $\wedge^m \mathrm{T}^*\mathcal{M}$, and thus splits $\wedge^m \mathrm{T}^*\mathcal{M}$ as a direct sum of the two eigen-subbundles $\wedge^m_\pm \mathrm{T}^*\mathcal{M}$ of $\star$. Elements of $\wedge^m_\pm \mathrm{T}^*\mathcal{M}$ are referred to as holomorphic *self-dual* and *anti-self-dual* m-forms.

This article is concerned with the local geometric properties of an *almost null structure* on $(\mathcal{M}, g)$, i.e. a holomorphic rank-m distribution $\mathcal{N} \subset \mathrm{T}\mathcal{M}$ totally null with respect to g, i.e. $g(v, w) = 0$ for all v and w in $\mathcal{N}_p$, and $\dim \mathcal{N}_p = m$ at any point p of $\mathcal{M}$. Being determined (i.e. annihilated) by a holomorphic m-form, an almost null structure may be either self-dual or anti-self-dual, and is also referred to as an *α-plane* or *β-plane distribution* accordingly.

There is a slick way to describe an almost null structure if we assume in addition $(\mathcal{M}, g)$ to be *spin*, i.e. it admits a holomorphic reduction to $\mathrm{Spin}(2m, \mathbb{C})$, the two-fold covering of $\mathrm{SO}(2m, \mathbb{C})$. In this case, $(\mathcal{M}, g)$ is endowed with two irreducible spinor bundles $\mathcal{S}^+$ and $\mathcal{S}^-$. Sections of $\mathrm{T}\mathcal{M}$ acts on sections of $\mathcal{S}^\pm$ via Clifford multiplication $\cdot : \mathrm{T}\mathcal{M} \times \mathcal{S}^\pm \to \mathcal{S}^\mp$. In particular, a holomorphic section ξ of $\mathcal{S}^+$ or $\mathcal{S}^-$ determines a distribution $\mathcal{N}_\xi$ on $\mathcal{M}$ in the sense that

$$(\mathcal{N}_\xi)_p := \{ v \in \mathrm{T}_p\mathcal{M} : v \cdot \xi_p \} \,, \qquad \text{at any point } p \text{ in } \mathcal{M}.$$

The defining property of the Clifford multiplication tells us that $\mathcal{N}_\xi$ is totally null. When $\mathcal{N}_\xi$ has dimension m at every point, ξ is said to be *pure*. If we refer to a pure spinor ξ defined *up to scale* as a *projective pure spinor* $[\xi]$, it is clear that a projective pure spinor field $[\xi]$ determines a unique almost null structure $\mathcal{N}_\xi$. Conversely, any almost null structure arises in this way. Whether ξ lies in $\mathcal{S}^+$ or $\mathcal{S}^-$ corresponds to whether $\mathcal{N}_\xi$ is self-dual or anti-self-dual. All spinors in $\mathcal{S}^\pm$ are pure in dimensions two, four and six, but when $m > 3$, the property of being pure imposes non-trivial algebraic conditions on the components of a spinor.

The geometric properties of an almost null structure $\mathcal{N}_\xi$ associated to a projective pure spinor $[\xi]$ can be expressed in terms of the covariant derivative of $[\xi]$. For instance, if $\mathcal{N}_\xi$ is integrable, i.e. $[\Gamma(\mathcal{N}_\xi), \Gamma(\mathcal{N}_\xi)] \subset \Gamma(\mathcal{N}_\xi)$, then one can show that the leaves of its foliation are totally geodetic, i.e. $\nabla_X Y \in \Gamma(\mathcal{N}_\xi)$ for any holomorphic sections X, Y of $\mathcal{N}_\xi$. This condition can also be expressed as [20]

$$\nabla_X \xi = \lambda_X \xi \,, \qquad \text{for any } X \in \Gamma(\mathcal{N}_\xi), \text{ and some holomorphic function } \lambda_X \text{ dependent on } X, \tag{1.1}$$

where, with a slight abuse of notation, ∇ denotes the spin connection induced from the Levi-Civita connection. Note that (1.1) is independent of the scale of ξ. Further, if ξ satisfies (1.1), then

$$C(X, Y, Z, W) = 0 \,, \qquad \text{for all } X, Y, Z, W \in \Gamma(\mathcal{N}_\xi). \tag{1.2}$$

where C denotes the Weyl tensor of ∇, i.e. the conformally invariant part of the Riemann tensor of ∇.

The investigation of conditions such as (1.1) and (1.2) will be the subject of this article. For this purpose, we note that an almost null structure $\mathcal{N}_\xi$ on $(\mathcal{M}, g)$ associated to a projective pure spinor field $[\xi]$ is equivalent to a holomorphic reduction of the structure group of $\mathrm{F}\mathcal{M}$ to the stabiliser $P \subset G := \mathrm{Spin}(2m, \mathbb{C})$ of $[\xi]$ at a point. This P is an instance of a *parabolic* subgroup, and is isomorphic to the semi-direct product $G_0 \ltimes P_+$ where part G_0 is reductive, and P_+ is nilpotent. The Lie algebras $\mathfrak{p} \subset \mathfrak{g} \cong \mathfrak{so}(2m, \mathbb{C})$ of P is isomorphic to $\mathfrak{g}_0 \oplus \mathfrak{p}_+$, where $\mathfrak{g}_0 \cong \mathfrak{gl}(m, \mathbb{C})$ and $\mathfrak{p}_+ \cong \wedge^2 \mathbb{C}^m$ are the Lie algebras of G_0 and P_+ respectively. Here, we have identified $(\mathcal{N}_\xi)_p \cong \mathbb{C}^m$ at any point p.