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New progress in the inverse problem in the calculus of variations

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A B S T R A C T

We present a new class of solutions for the inverse problem in the calculus of variations in arbitrary dimension n. This is the problem of determining the existence and uniqueness of Lagrangians for systems of n second order ordinary differential equations. We also provide a number of new theorems concerning the inverse problem using exterior differential systems theory (EDS). Concentrating on the differential step of the EDS process, our new results provide a significant advance in the understanding of the inverse problem in arbitrary dimension. In particular, we indicate how to generalise Jesse Douglas's famous solution for $n = 2$. We give some non-trivial examples in dimensions 2, 3 and 4. We finish with a new classification scheme for the inverse problem in arbitrary dimension.

1. Introduction: the inverse problem

The inverse problem in the calculus of variations can be expressed as follows. Given a system of second-order ordinary differential equations

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b), \quad a, b = 1, \ldots, n,$$

the question is whether the solutions of this system are also the solutions of the Euler Lagrange equations,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}^a}\Big) - \frac{\partial L}{\partial x^a} = 0,$$

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for some regular Lagrangian $L(t, x^b, \dot{x}^b)$. This problem was first proposed by Helmholtz [11] in 1887. He considered whether the equations *in the form presented* were Euler–Lagrange. In the case of single equations Helmholtz found necessary conditions for this to be true. It is not well-known that Sonin [23] solved this one-dimensional problem the previous year in a more general form, although Hirsch [12] is credited with the so-called multiplier version of the inverse problem, which is the focus of this paper. He addressed the uniqueness and existence of a non-degenerate multiplier matrix, $g_{ab}(t, x^c, \dot{x}^c)$, satisfying

$$g_{ab}(\ddot{x}^b - F^b) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a}.$$

Necessary and sufficient conditions for the existence of a regular Lagrangian, according to Douglas [9] and to Sarlet [19], are that this multiplier satisfies

$$g_{ab} = g_{ba}, \quad \Gamma(g_{ab}) = g_{ac}\Gamma^c_b + g_{bc}\Gamma^c_a, \quad g_{ac}\Phi^c_b = g_{bc}\Phi^c_a, \quad \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b}, \tag{1}$$

where

$$\Gamma^a_b := -\frac{1}{2}\frac{\partial F^a}{\partial \dot{x}^b}, \quad \Phi^a_b := -\frac{\partial F^a}{\partial x^b} - \Gamma^c_b\Gamma^a_c - \Gamma(\Gamma^a_b),$$

and where

$$\Gamma := \frac{\partial}{\partial t} + \dot{x}^a\frac{\partial}{\partial x^a} + F^a\frac{\partial}{\partial \dot{x}^a}.$$

These conditions, along with non-degeneracy, are commonly referred to as the *Helmholtz conditions*. If one or more matrices g_{ab} are found that satisfy these four Helmholtz conditions, then there exists one (or more) Lagrangians related to these matrices by the expression,

$$\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b} = g_{ab}.$$

Integrating this for a given g_{ab} we see that the related Lagrangian L is only defined up to the addition of a total time derivative of an arbitrary function of t, x^a.

The multiplier problem was completely solved by Douglas in 1941 for the two dimensional case (see [9]), that is, a pair of second order equations on the plane. He divided the problem into four primary cases (I to IV) according to the properties of the matrix Φ^a_b. The corresponding solution for higher dimensions, even for dimension 3, remained unsolved until the late nineteen nineties when some arbitrary n subcases were elaborated [5,20,3].

The current attacks on this problem, dating back to the 1980s, involve the creation and use of various differential geometric tools. We offer the reader the following references which give some perspective on these developments [3,6,7,10,14,16,17,21].

The current situation, in the framework of [7], is that the following dimension n situations are solved in the sense of Douglas.

1. Φ is a multiple of the identity. The multiplier is determined by n arbitrary functions each of $n+1$ variables. This is the extension of Douglas's case I. See [2,3,20].
2. Φ is diagonalisable with distinct eigenvalues and "integrable eigenspaces". The multiplier is determined by n arbitrary functions each of 2 variables. This is the extension of Douglas's case IIa1. See [5,1].
3. There are many non-existence results depending on technical conditions on Φ. See [18].