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On the Gauss–Bonnet–Chern formula for real Finsler vector bundles

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ABSTRACT

In this paper, we give a Gauss–Bonnet–Chern formula for real Finsler vector bundles with respect to any metric-compatible connection. The key idea is to modify any given metric-compatible connection to be a new metric-compatible connection with some special properties. As a corollary, a Gauss–Bonnet–Chern formula for Finsler manifolds with respect to any metric-compatible connection is established.

1. Introduction

A real Finsler vector bundle $(\mathscr{E}, F, M)$ is a real vector bundle $\mathscr{E}$ of rank n over an m-dimensional differentiable manifold M, equipped with a Finsler metric F, which is a nonnegative function on $\mathscr{E}$ satisfying the following conditions:

(i) F is smooth on the slit bundle $\mathscr{E}\backslash 0$;

(ii) F is positively homogeneous, i.e., $F(\lambda y) = \lambda F(y)$, for all $\lambda > 0$, $y \in \mathscr{E}$;

(iii) The Hessian $[F^2]_{y^i y^j}(x, y)$ is positive definite, where $F(x, y) := F(y^i s_i|_x)$ and $\{s_i\}$ is a local frame field of $\mathscr{E}$.

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Clearly, real Finsler vector bundles are more general than Riemannian vector bundles. Furthermore, the tangent bundle of a Finsler manifold is a real Finsler vector bundle, whose horizontal bundle and vertical bundle are isomorphic to each other (cf. [3]).

The main purpose of this paper is to establish a Gauss–Bonnet–Chern (GBC) formula for real Finsler vector bundles with respect to any metric-compatible connection. More precisely, given an oriented real Finsler vector bundle $(\mathscr{E}, F)$ of rank n over an n-dimensional compact oriented closed manifold M, let $\pi : S\mathscr{E} \to M$ be the projective sphere bundle and let $\pi^*\mathscr{E}$ be the pull-back bundle. The Finsler metric F then induces a Riemannian metric g on $\pi^*\mathscr{E}$. An operator D is called a metric-compatible connection of $(\mathscr{E}, F)$, if it is a linear connection on $\pi^*\mathscr{E}$ metric-compatible with g. Given a metric-compatible connection D of $(\mathscr{E}, F)$, define the n-form $\mathbf{\Omega}^D$ as follows:

$$\mathbf{\Omega}^D = \begin{cases} \frac{(-1)^p}{2^{2p}\pi^p p!}\epsilon_{i_1\ldots i_{2p}}\Omega^{i_2}_{i_1} \wedge \cdots \wedge \Omega^{i_{2p}}_{i_{2p-1}}, & n = 2p, \\ 0, & n = 2p+1, \end{cases} \tag{1.1}$$

where (Ω^i_j) is the local curvature 2-form of D, $\epsilon_{i_1\ldots i_{2p}}$ is the multiple Kronecker Delta. Then, by using a new approach, we shall prove the following

Theorem 1.1. *Let $\mathscr{E}$ be an oriented real Finsler vector bundle of rank n over an n-dimensional closed oriented manifold M. Given any metric-compatible connection D, for any smooth section X with isolated zeros on $\mathscr{E}$, we have*

$$\int_M [X]^* \left(\frac{\mathbf{\Omega}^D + \mathfrak{E}}{\mathrm{V}(x)} \right) = \frac{\chi(\mathscr{E})}{\mathrm{vol}(\mathbb{S}^{n-1})}, \tag{1.2}$$

where $[X]$ is the section of $S\mathscr{E}$ induced by X, $\mathrm{V}(x)$ is the Riemannian volume of $\pi^{-1}(x)$, $\mathfrak{E}$ is an n-form on $S\mathscr{E}$ and $\chi(\mathscr{E})$ is the Euler characteristic of $\mathscr{E}$.

In order to describe Theorem 1.1 in more detail, we recall some background material. It is well known that seventy years ago, S.S. Chern [9,10] gave an intrinsic proof of the GBC theorem for all oriented compact closed n-dimensional Riemannian manifolds (M, g), that is,

$$\int_M \mathbf{\Omega} = \chi(M), \tag{1.3}$$

where $\mathbf{\Omega}$ is defined as in (1.1) with respect to the Levi-Civita connection, $\chi(M)$ is the Euler characteristic of M. And the Chern–Weil theory then yields (1.3) exactly holds for all metric-compatible connections (cf. [6,16,20]). Moreover, (1.3) has recently been generalized to Riemannian vector bundles (cf. [5,17]), which reveals an intrinsic fact: the integral of the geometric Euler class is exactly the Euler characteristic in the Riemannian case. It is noticeable that Riemannian bundles and the tangent bundles of Riemannian manifolds are real Finsler bundles and (hence,) all these results can be derived from Theorem 1.1. Specifically, Theorem 1.1 yields the following two GBC theorems:

Theorem 1.2. *(See [6,16].) Let (M, g) be an closed compact oriented Riemannian manifold. Given any metric-compatible connection $\mathcal{D}$, we have*

$$\int_M \mathbf{\Omega}^{\mathcal{D}} = \chi(M).$$