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On the total curvature and extrinsic area growth of surfaces with tamed second fundamental form ☆

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A B S T R A C T

In this paper we show that a complete and non-compact surface immersed in the Euclidean space with quadratic extrinsic area growth has finite total curvature provided the surface has tamed second fundamental form and admits total curvature. In such a case we obtain as well a generalized Chern–Osserman inequality. In the particular case of a surface of nonnegative curvature, we prove that the surface is diffeomorphic to the Euclidean plane if the surface has tamed second fundamental form, and that the surface is isometric to the Euclidean plane if the surface has strongly tamed second fundamental form. In the last part of the paper we characterize the fundamental tone of any submanifold of tamed second fundamental form immersed in an ambient space with a pole and quadratic decay of the radial sectional curvatures.

1. Introduction

Let M be a complete non-compact surface and let K denote the Gaussian curvature of M. We will say that M admits total curvature $\int_M K\,dA$ (finite or infinite) if for any compact exhaustion $\{\Omega_i\}$ of M, the limit

$$\int_M K\,dA = \lim_{i\to\infty} \int_{\Omega_i} K\,dA$$

exists. Cohn-Vossen proved in [10] that $\int_M K\,dA \leq \chi(M)$, where $\chi(M)$ is the Euler characteristic of M. A well known theorem due to Huber [18] states that if the negative part of the curvature $K_- = \max\{-K, 0\}$ has finite integral, namely,

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$$\int_M K_-\, dA < \infty, \tag{1.1}$$

then, $\int_M K\, dA \leq \chi(M)$ and M is conformally equivalent to a compact Riemann surface with finitely many punctures. Hartman, under the assumption (1.1) proved in [17] that the area $\mathrm{A}(B_r)$ of a geodesic ball of radius r at a fixed point must grow at most quadratically in r. Reciprocally, Li proved in [23] that if M has at most quadratic area growth, finite topology and if the Gaussian curvature of M is either non-positive or non-negative, near infinity of each end, then M must have finite total curvature.

From an extrinsic point of view, in the setting of a minimal surface M immersed in the Euclidean space $\mathbb{R}^n$, it is well known, see [9,20,25,26], that if M has finite total curvature then M has finite topological type and *quadratic extrinsic area growth*, *i.e.*, there exists a constant C such that for any $r \in \mathbb{R}_+$

$$\mathrm{Area}(M \cap B_r(0)) \leq Cr^2, \tag{1.2}$$

where $B_r(0)$ denotes the geodesic ball centered at the origin $0 \in \mathbb{R}^n$ of radius r.

Conversely, Q. Chen [8], proved that if M is an oriented complete minimal surface in the Euclidean space $\mathbb{R}^n$ with quadratic extrinsic area growth and finite topological type then M has finite total curvature.

A natural question is whether an equivalent result relating the extrinsic area growth and the total curvature holds for a broader class of complete surfaces in the Euclidean space. The aim of this paper is to provide an answer to this question under certain control of the second fundamental form of the immersion. A surface M is said to have tamed second fundamental form if for a (any) compact exhaustion $\{\Omega_i\}$ of M,

$$a(M) := \lim_{i\to\infty}\left(\sup_{x\in M\setminus\Omega_i}\{\rho_M(x)\|\alpha(x)\|\}\right) < 1, \tag{1.3}$$

where $\rho_M(x) = \mathrm{dist}_M(x_0, x)$ is the distance function on M to a fixed point x_0, and $\|\alpha(x)\|$ is the norm of the second fundamental form at $\varphi(x)$. The notion of immersion with tamed second fundamental form was introduced in [5] for submanifolds of $\mathbb{R}^n$ and in [4] for submanifolds of Hadamard manifolds. This notion can be naturally extended to manifolds with a pole and radial sectional curvature bounded above, see [13]. Under the hypothesis of tamed second fundamental form and quadratic extrinsic area growth we can state the following result.

Theorem 1.1. *Let M be an immersed complete oriented surface of $\mathbb{R}^n$ with curvature function K and tamed second fundamental form. Suppose that M admits total curvature. Then, M has finite total curvature ($\int_M KdA > -\infty$), if and only if, M has quadratic extrinsic area growth,* i.e., *there exists a constant C_1 such that,*

$$A(M \cap B_r(0)) \leq C_1 r^2, \tag{1.4}$$

for any r large enough. Furthermore, if (1.4) holds, then there exists a constant $C_0 > 0$ such that

$$A(M \cap B_r(0)) \geq C_0 r^2, \tag{1.5}$$

for any r large enough.

Observe that the assumption that the surface admits total curvature (finite or infinite) can be achieved if the surface has semidefinite curvature (either nonpositive or nonnegative). As observed by Jorge–Meeks [20], any complete m-dimensional submanifold M of $\mathbb{R}^n$ homeomorphic to a compact Riemannian manifold $\overline{M}$ punctured at a finite number of points $\{p_1, \ldots, p_r\}$ and having a well defined normal vector at