Differential Geometry and its Applications 47 (2016) 99–129

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

On diameter controls and smooth convergence away from singularities

Sajjad Lakzian [1]

Fordham University, United States

ARTICLE INFO

Article history:
Received 13 January 2014
Available online 4 April 2016
Communicated by F. Pedit

MSC:
53C20
51Fxx

Keywords:
Intrinsic Flat
Ricci Flow

ABSTRACT

We prove that if a family of metrics, g_i, on a compact Riemannian manifold, M^n, have a uniform lower Ricci curvature bound and converge to g_∞ smoothly away from a singular set, S, with Hausdorff measure, $H^{n-1}(S) = 0$, and if there exists connected precompact exhaustion, W_j, of $M^n \setminus S$ satisfying $\mathrm{diam}_{g_i}(M^n) \le D_0$, $\mathrm{Vol}_{g_i}(\partial W_j) \le A_0$ and $\mathrm{Vol}_{g_i}(M^n \setminus W_j) \le V_j$ where $\lim_{j\to\infty} V_j = 0$ then the Gromov–Hausdorff limit exists and agrees with the metric completion of $(M^n \setminus S, g_\infty)$. This is a strong improvement over prior work of the author with Sormani that had the additional assumption that the singular set had to be a smooth submanifold of codimension two. We have a second main theorem in which the Hausdorff measure condition on S is replaced by diameter estimates on the connected components of the boundary of the exhaustion, ∂W_j. This second theorem allows for singular sets which are open subregions of the manifold. In addition, we show that the uniform lower Ricci curvature bounds in these theorems can be replaced by the existence of a uniform linear contractibility function. If this condition is removed altogether, then we prove that $\lim_{j\to\infty} d_{\mathcal{F}}(M'_j, N') = 0$, in which M'_j and N' are the settled completions of (M, g_j) and $(M_\infty \setminus S, g_\infty)$ respectively and $d_{\mathcal{F}}$ is the Sormani–Wenger Intrinsic Flat distance. We present examples demonstrating the necessity of many of the hypotheses in our theorems.

1. Introduction

In this paper, our goal is to deepen our understanding about smooth convergence of Riemannian metrics away from singular sets in a very general setting and give a complete list of our theorems for reference. We have also included several examples and remarks in order to help the reader to understand how these notions of convergence interplay.

For us, smooth convergence away from singular set S is defined as follows:

E-mail address: slakzian@fordham.edu.

[1] Lakzian was partially supported by NSF DMS #1006059.

http://dx.doi.org/10.1016/j.difgeo.2016.01.003
0926-2245/

Definition 1.1. A sequence of Riemannian metrics g_i on a compact manifold M^n is said to converge smoothly away from $S \subset M^n$ to a Riemannian metric g_∞ on $M^n \setminus S$ if for every compact set $K \subset M^n \setminus S$, g_i converge $C^{k,\alpha}$ smoothly to g_∞ as tensors.

Even though the Definition 1.1 makes sense for all $k \geq 0$ and $0 < \alpha < 1$, we will always assume that the convergence is in C^∞ or at least $C^{2,\alpha}$ (the $C^{2,\alpha}$ regularity is needed later on, when we assume Ricci curvature bounds).

Right away from the definition, it is apparent that the global geometry is not well controlled under such convergence. It is natural to ask under what additional conditions the original sequence of manifolds, $M_i = (M^n, g_i)$ has the expected Gromov–Hausdorff (GH) and Sormani–Wenger Intrinsic Flat (SWIF) limits [10,20]. Recall that there are examples of sequences of metrics on spheres which converge smoothly away from a point singularity which have no subsequence converging in the GH or the SWIF sense, so additional conditions are necessary (cf. [13]).

In the past decades, many results have appeared dealing with singular Gromov–Hausdorff limit of Riemannian metrics. To mention a few important ones, Anderson in [2] studies the convergence of 4-dimensional Einstein manifolds to orbifolds. Bando–Kasue–Nakajima in [4] study the singularities of the Einstein ALF manifolds. Eyssidieux–Guedj–Zeriahi in [7] prove similar results for the solutions to the complex Monge–Ampere equation. Also Huang in [11], Ruan Zhang in [17], Sesum in [18], Tian in [21] and Tosatti in [23] study the convergence of Kahler–Einstein metrics and Kahler–Einstein orbifolds. However, even in this setting, the relationship is not completely clear and the limits need not agree (see [3]). In Tian and Viaclovsky [22], compactness results for various classes Riemannian metrics in dimension four were obtained in particular for anti-self-dual metrics, Kahler metrics with constant scalar curvature, and metrics with harmonic curvature. Also the relation between different notions of convergence for Ricci flow is studied in [12].

Here we first study SWIF limits of sequences of manifolds which converge away from a singular set and then prove the SWIF and GH limits agree using techniques developed in prior work of the author with Sormani in [13]. All necessary background on these techniques and on SWIF convergence is reviewed in Section 2.

Theorem 1.2. *Let $M_i = (M^n, g_i)$ be a sequence of compact oriented Riemannian manifolds such that there is a subset, S, with $H^{n-1}(S) = 0$ and connected precompact exhaustion, W_j, of $M \setminus S$ satisfying (8) with g_i converge smoothly to g_∞ on each W_j,*

$$\operatorname{diam}_{M_i}(W_j) \leq D_0 \qquad \forall i \geq j, \tag{1}$$

$$\operatorname{Vol}_{g_i}(\partial W_j) \leq A_0, \tag{2}$$

and

$$\operatorname{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j\to\infty} V_j = 0. \tag{3}$$

Then

$$\lim_{i\to\infty} d_{\mathcal{F}}(M_i', N') = 0, \tag{4}$$

where M_i' and N' are the settled completion of (M, g_i) and $(M \setminus S, g_\infty)$ respectively.

Here $\operatorname{diam}_M(W)$ is the extrinsic diameter found by

$$\operatorname{diam}_M(W) = \sup\{d_M(x, y) : x, y \in W\} \tag{5}$$