Differential Geometry and its Applications 47 (2016) 159–189

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Stabilities of affine Legendrian submanifolds and their moduli spaces

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ARTICLE INFO

Article history:
Received 7 September 2015
Received in revised form 22 December 2015
Available online 19 April 2016
Communicated by V. Cortes

MSC:
53C42
53C25

Keywords:
Affine Lagrangian submanifolds
Affine Legendrian submanifolds
Stabilities
Moduli spaces

ABSTRACT

We introduce the notion of affine Legendrian submanifolds in Sasakian manifolds and define a canonical volume called the ϕ-volume as odd dimensional analogues of affine Lagrangian (totally real or purely real) geometry. Then we derive the second variation formula of the ϕ-volume to obtain the stability result in some η-Einstein Sasakian manifolds. It also implies the convexity of the ϕ-volume functional on the space of affine Legendrian submanifolds.

Next, we introduce the notion of special affine Legendrian submanifolds in Sasaki–Einstein manifolds as a generalization of that of special Legendrian submanifolds. Then we show that the moduli space of compact connected special affine Legendrian submanifolds is a smooth Fréchet manifold.

1. Introduction

Affine Lagrangian (totally real or purely real) submanifolds are “maximally non-complex” submanifolds in almost complex manifolds defined by relaxing the Lagrangian condition (Definition 3.1). The affine Lagrangian condition is an open condition and hence there are many examples. Borrelli [2] defined a canonical volume of an affine Lagrangian submanifold called the J-volume. He obtained the stability result for the J-volume as in the Lagrangian case [4]. Lotay and Pacini [8] pointed out the importance of affine Lagrangian submanifolds in the study of geometric flows. Opozda [11] showed that the moduli space of (special) affine Lagrangian submanifolds was a smooth Fréchet manifold.

In this paper, we study the odd dimensional analogue. First, we introduce the notion of affine Legendrian submanifolds in Sasakian manifolds and define a canonical volume called the ϕ-volume as odd dimensional analogues of affine Lagrangian geometry. See Definitions 3.8 and 3.12. Then we compute the first variation of the ϕ-volume and characterize a critical point for the ϕ-volume by the vanishing of some vector field H_ϕ

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http://dx.doi.org/10.1016/j.difgeo.2016.04.001

(Proposition 4.5), which is a generalization of the mean curvature vector (Remark 4.6). We call an affine Legendrian submanifold ϕ**-minimal** if $H_\phi = 0$. Then we compute the second variation of the ϕ-volume and obtain the following.

Theorem 1.1. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$-dimensional Sasakian manifold and $\iota : L^n \hookrightarrow M$ be an affine Legendrian immersion of a compact oriented n-dimensional manifold L. Let $\iota_t : L \hookrightarrow M$ be a one-parameter family of affine Legendrian immersions satisfying $\iota_0 = \iota$. Suppose that $\frac{\partial \iota_t}{\partial t}|_{t=0} = Z = \phi Y + f\xi$, where $Y \in \mathfrak{X}(L)$ is a vector field on L and $f \in C^\infty(L)$ is a smooth function. Then we have*

$$\frac{d^2}{dt^2} \int_L \mathrm{vol}_\phi[\iota_t]\Bigg|_{t=0} = \int_L \Bigg((2n+2)\eta(Y)^2 - 2g(Y,Y) - \mathrm{Ric}(Y,Y) - g(\pi_L[Z,Y], H_\phi) + g(Y, H_\phi)^2 + \left(\frac{\mathrm{div}(\rho_\phi[\iota]Y)}{\rho_\phi[\iota]}\right)^2 \Bigg) \mathrm{vol}_\phi[\iota],$$

where $\mathrm{vol}_\phi[\iota]$ is the ϕ-volume form of ι given in Definition 3.12, Ric *is the Ricci curvature of (M, g), $\pi_L : \iota^* TM \to \iota_* TL$ is the canonical projection given in (4), $\rho_\phi[\iota]$ is the function on L given in Definition 3.12 and H_ϕ is the vector field on L given in Definition 4.3.*

Remark 1.2. For Legendrian submanifolds, the ϕ-volume agrees with the standard Riemannian volume (Lemma 3.14). When ι is minimal Legendrian and all of ι_t's are Legendrian, Theorem 1.1 agrees with [10, Theorem 1.1]. When ι is Legendrian-minimal Legendrian and all of ι_t's are Legendrian, Theorem 1.1 agrees with [7, Theorem 1.1]. See Remark 5.3.

Following the Riemannian case, we call a ϕ-minimal affine Legendrian submanifold ϕ**-stable** if the second variation of the ϕ-volume is nonnegative.

Now, suppose that a $(2n+1)$-dimensional Sasakian manifold $(M^{2n+1}, g, \eta, \xi, \phi)$ is a η-Einstein with the η-Ricci constant $A \in \mathbb{R}$. (See Definition 2.5.) Then we obtain the following.

Theorem 1.3. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$-dimensional η-Einstein Sasakian manifold with the η-Ricci constant $A \le -2$. Then any ϕ-minimal affine Legendrian submanifold in M is ϕ-stable.*

This is a generalization of [10, Theorem 1.2]. The author obtained further results by restricting variations of a minimal Legendrian submanifold to its Legendrian variations. In our case, since the affine Legendrian condition is an open condition, any small variation is affine Legendrian. Thus there is no analogue of these results.

Similarly, using the notion of convexity in the space of affine Legendrian submanifolds (Definition 3.17), we easily see the following.

Theorem 1.4. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$-dimensional η-Einstein Sasakian manifold with the η-Ricci constant $A \le -2$. Then the ϕ-volume functional on the space of affine Legendrian submanifolds is convex.*

For affine Legendrian submanifolds in a η-Einstein Sasakian manifold with the η-Ricci constant $A > -2$, we have the following.

Theorem 1.5. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$-dimensional η-Einstein Sasakian manifold with the η-Ricci constant $A > -2$. Then there are no ϕ-minimal affine Legendrian submanifolds which are ϕ-stable.*