



An alternative proof of a theorem of Davis and Fang



Yueshan Xiong

Department of Mathematics, Capital Normal University, Beijing 100048, China

ARTICLE INFO

Article history:

Received 2 November 2015
Received in revised form 29 May 2016

Available online 17 June 2016
Communicated by F. Fang

MSC:

57R19
57R20
55N91

Keywords:

Almost flat manifold
Infranilmanifold
Equivariant bundle

ABSTRACT

It is conjectured that every almost flat manifold bounds a compact manifold. Davis and Fang proved that every infranilmanifold with cyclic or generalized quaternionic holonomy bounds a compact manifold. In this paper, we give an alternative proof of this theorem by using equivariant topology.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

Farrell and Zdravkovska [4] and independently S.T. Yau [12] conjectured that an almost flat manifold bounds a compact manifold. This conjecture is still open. By the results of Gromov and Ruh [6,10], every almost flat manifold is diffeomorphic to an infranilmanifold which is a double coset space $\Gamma \backslash L \rtimes G/G$, where L is a simply connected nilpotent Lie group, G is a finite subgroup of $\text{Aut}(L)$ and Γ is a discrete torsion-free cocompact subgroup of $L \rtimes G$ which maps epimorphically to G under the projection $L \rtimes G \rightarrow G$. The finite group G is called the holonomy group of $\Gamma \backslash L \rtimes G/G$. Therefore, we only need to consider this conjecture for infranilmanifolds.

Hamrick and Royster [7] proved the conjecture for flat manifolds. For almost flat manifolds, there are some results. Farrell and Zdravkovska [4] proved that the conjecture is true when the holonomy group G is \mathbb{Z}_2 or G acts effectively on the center of L . Upadhyay [11] proved the conjecture when L is a simply connected 2-step nilmanifold, the holonomy group G is cyclic and G acts trivially on the center of L . The most recent progress is from Davis and Fang [3] who showed that the conjecture is true when the holonomy group G is a cyclic group or a generalized quaternion group.

E-mail address: yueshan_xiong@yahoo.com.

In this paper, we give an alternative proof of the following Theorem of Davis and Fang [3].

Theorem 1.1. [3] *Let M be an infranilmanifold and $\text{Syl}_2 G$ be the 2-sylow subgroup of its holonomy group. If $\text{Syl}_2 G$ is cyclic or generalized quaternionic, then M is the boundary of a compact manifold.*

In the following theorem, we prove the conjecture for a special case.

Theorem 1.2. *Let $M = \Gamma \backslash L \rtimes G/G$ be an infranilmanifold. If L is a simply connected 2-step nilpotent Lie group with $\dim[L, L] = 1$ and G acts trivially on the center of L , then M is the boundary of a compact manifold.*

In this paper, we only consider the cohomology with \mathbb{Z}_2 -coefficients, which is denoted by $H^*(X)$ for a space X .

2. Some technical lemmas

2.1. Some properties of equivariant cohomology of G spaces

In this subsection, we give several lemmas on the equivariant cohomology of G -spaces. Denote $X_G = EG \times_G X$ for any G space X .

Lemma 2.1. *If X is a CW complex with a cellular $G \times \mathbb{Z}_2$ -action such that the subgroups $G \times e$ and $e \times \mathbb{Z}_2$ act freely, and if $\overline{X} = X/\mathbb{Z}_2$ and $\overline{X}_G = EG \times_G \overline{X}$, then there is an exact sequence*

$$\cdots \longrightarrow H^i(\overline{X}_G) \xrightarrow{\cup \omega_1} H^{i+1}(\overline{X}_G) \longrightarrow H^{i+1}(X/G) \longrightarrow H^{i+1}(\overline{X}_G) \longrightarrow \cdots \quad (1)$$

with mod 2 coefficients which is also natural with respect to the equivariant maps in X .

Proof. For any CW complex X with a $G \times \mathbb{Z}_2$ action, there is an action of \mathbb{Z}_2 on $EG \times_G X$ defined by $[u, x] \cdot \alpha = [u, x \cdot \alpha]$, where $[u, x] \in EG \times_G X$ and $\alpha \in \mathbb{Z}_2$. If there exist an element $[u, x] \in EG \times_G X$ and an element $\alpha \in \mathbb{Z}_2$ such that $[u, x] \cdot \alpha = [u, x \cdot \alpha] = [u, x]$, then we have $x \cdot \alpha = x$. Therefore, if $G \times e$ and $e \times \mathbb{Z}_2$ act freely on X , then the action of \mathbb{Z}_2 on $EG \times_G X$ is free. The quotient space of $EG \times_G X$ by the action of \mathbb{Z}_2 is $EG \times_G \overline{X}$, where G acts on \overline{X} via $\overline{x} \cdot g = \overline{x \cdot g}$. So we have a \mathbb{Z}_2 covering $\tilde{p}: EG \times_G X \rightarrow EG \times_G \overline{X}$. By the Gysin sequence [9, Corollary 12.3], there is an exact sequence

$$\cdots \longrightarrow H^i(\overline{X}_G) \xrightarrow{\cup \omega_1} H^{i+1}(\overline{X}_G) \xrightarrow{\tilde{p}^*} H^{i+1}(X_G) \xrightarrow{\delta} H^{i+1}(\overline{X}_G) \longrightarrow \cdots,$$

where ω_1 is the first Stiefel–Whitney class of the covering $\tilde{p}: EG \times_G X \rightarrow EG \times_G \overline{X}$. Since G acts freely on X , the projection map $\pi_2: EG \times_G X \rightarrow X/G$ is a homotopy equivalence. Thus $\pi_2^*: H^*(X/G) \rightarrow H^*(X_G)$ is an isomorphism. By replacing $H^*(X_G)$ by $H^*(X/G)$ in the above exact sequence, we get the exact sequence (1).

Let X, Y be CW complexes with $G \times \mathbb{Z}_2$ actions satisfying the conditions of the lemma and $h: X \rightarrow Y$ be an equivariant map. By the definition of the actions of \mathbb{Z}_2 on $EG \times_G X$ and $EG \times_G Y$, we have the commutative diagrams

$$\begin{array}{ccc} EG \times_G X & \xrightarrow{Id \times_G h} & EG \times_G Y \\ \tilde{p} \downarrow & & \downarrow \tilde{p}' \\ EG \times_G \overline{X} & \xrightarrow{Id \times_G \overline{h}} & EG \times_G \overline{Y} \end{array} \qquad \begin{array}{ccc} EG \times_G X & \xrightarrow{Id \times_G h} & EG \times_G Y \\ \pi_2 \downarrow & & \downarrow \pi'_2 \\ X/G & \xrightarrow{h'} & Y/G, \end{array}$$

Download English Version:

<https://daneshyari.com/en/article/4605819>

Download Persian Version:

<https://daneshyari.com/article/4605819>

[Daneshyari.com](https://daneshyari.com)