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Kirillov structures up to homotopy

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ABSTRACT

We present the notion of *higher Kirillov brackets* on the sections of an even line bundle over a supermanifold. When the line bundle is trivial we shall speak of *higher Jacobi brackets*. These brackets are understood furnishing the module of sections with an L_∞-algebra, which we refer to as a *homotopy Kirillov algebra*. We are then led to *higher Kirillov algebroids* as higher generalisations of Jacobi algebroids. Furthermore, we show how to associate a higher Kirillov algebroid and a homotopy BV-algebra with every higher Kirillov manifold. In short, we construct homotopy versions of some of the well-known theorems related to Kirillov's local Lie algebras.

1. Introduction

The rôle of graded geometries beyond those found in supersymmetry in physics has emerged since the 1980s with the development of the BV-formalism and its refinements and generalisations. Mathematically all this fits under Stasheff's notion of 'cohomological physics': that is the identification and study of structures in physics that have their true mathematical understanding in homological algebra and homotopy theory. The pinnacle of cohomological physics has to be the development of homotopy-coherent structures, such the well-known A_∞ and L_∞-algebras, and their applications in the BV-formalism, the BFV-formalism, string field theory and deformation quantisation. The abundance of brackets appearing in quantum field theory and the relations between them were described by Huebschmann [16] as the *bracket zoo*.

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Focussing in on the L_∞-algebras, an interesting class of such structures are the homotopy Poisson algebras; see for example [4–6,9,17,27,28,35]. Homotopy Poisson algebras appear naturally in the geometry of coisotropic submanifolds of Poisson manifolds. Specifically they play a relevant rôle in homological Poisson reduction [31], deformation and moduli theory [28,30], and deformation quantisation [9] of coisotropic submanifolds. Loosely a homotopy Poisson algebra is an *L_∞-algebra structure, suitably graded such that the series of brackets satisfy a Leibniz rule over a graded commutative product*.

Jacobi manifolds [25] and the related local Lie algebras/Jacobi bundles [19,26] (which are referred to as abstract Jacobi manifolds in [23]) were originally introduced as generalisations of Poisson manifolds that in a sense 'interpolate' between symplectic and contact manifolds. On the other hand Jacobi manifolds can be seen as a specialisation of Poisson structures via the 'Poissonisation' process.

A natural question that we pose and answer is *can one develop a theory of higher or homotopy Jacobi structures and the associated L_∞-algebras?* This question was originally asked by the first author in [7]. A related notion is that of a *generalised Jacobi structure*, as defined by Pérez Bueno [29]. The problems with this notion are twofold. First, although the links with L_∞-algebras are there, the structure is concentrated into a single bracket. The relation with homotopy theory is thus obscured. Secondly, the approach relies on the line bundle being trivial: we know that the correct and full understanding of Jacobi structures is via Kirillov's picture of local Lie algebras [19].

L_∞-algebras appear naturally in the geometry of coisotropic submanifolds of Jacobi manifolds. Specifically, as described in [23] and [24], they play a relevant rôle in homological Jacobi reduction, and deformation and moduli theory of coisotropic submanifolds. It is not hard to imagine that homotopy Kirillov algebras may well play a rôle in the deformation quantisation of coisotropic submanifolds in the contact and Jacobi setting. From this perspective, bracket structures on line bundles seem worthy of study, and form part of a wider understanding of 'brackets' on supermanifolds – which include L_∞-algebroids (cf. [6,17]).

The purely algebraic generalisation of homotopy Jacobi algebras and their associated homotopy BV-algebras were studied by Vitagliano [34]. We also comment that Vitagliano has developed a notion of multicontact forms [33], and again L_∞-algebras appear quite naturally there.

Kirillov Manifolds: Let us concentrate briefly on the purely even case for a moment. Recall that one can identify smooth sections of a line bundle L with smooth homogeneous functions of degree one on the dual line bundle L^* and further also with homogeneous functions of degree one on the principal $\mathbb{R}^\times$-bundle $(L^*)^\times := L^* \setminus \{\underline{0}\}$, i.e. functions $f : (L^*)^\times \to \mathbb{R}$ such that $f(\mathrm{h}_t(x)) := f(t \cdot x) = tf(x)$, where h is the action of $\mathbb{R}^\times$. Let us denote this identification as $u \rightsquigarrow \iota_u$, where $u \in \mathsf{Sec}(L)$. This identification allows for a very useful characterisation of Kirillov brackets in terms of *Kirillov manifolds*.

Definition 1.1 *(Grabowski [15]).* A *principal Poisson $\mathbb{R}^\times$-bundle*, shortly *Kirillov manifold*, is a principal $\mathbb{R}^\times$-bundle (P, h) equipped with a Poisson structure Λ of degree -1, i.e. such that $(\mathrm{h}_s)_*\Lambda = s^{-1}\Lambda$.

Theorem 1.1 *(Grabowski [15]). There is a one-to-one correspondence between Kirillov brackets $[\cdot, \cdot]_L$ on a line bundle $L \to M$ and Poisson structures Λ of degree -1 on the principal $\mathbb{R}^\times$-bundle $P = (L^*)^\times$ given by*

$$\iota_{[u,v]_L} = \{\iota_u, \iota_v\}_\Lambda.$$

If the line bundle in question is trivial, then we have a *Jacobi manifold*. With the above theorem in mind, Jacobi manifolds and Jacobi bundles can be seen as *specialisations* and not only as generalisations of Poisson manifolds. If the homogeneous Poisson structure is non-degenerate, that is associated with a symplectic form, then we are discussing contact geometry. We will take the description of Jacobi geometry in terms of Kirillov manifolds as the fundamental definition. This point of view is further pursued in [8] and applied to the theory of contact groupoids. Of course the link between Jacobi structures and homogeneous