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On a class of projectively flat Finsler metrics ☆

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In this paper, we classify locally projectively flat general (α,β)-metrics $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ on an $n(\geq 3)$-dimensional manifold if α is of constant sectional curvature and $\phi_1 \neq 0$. Furthermore, we find equations to characterize this class of metrics with constant flag curvature and determine their local structures.

1. Introduction

The regular case of the Hilbert's Fourth Problem is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose positive geodesics are straight lines. Such Finsler metrics are called *locally projectively flat* Finsler metrics in $\mathbb{R}^n$. This problem has been solved in Riemannian geometry by Beltrami, which says that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. However, Beltrami's theorem is not true in Finsler geometry. There are Finsler metrics of scalar (resp. constant) flag curvature which are not locally projectively flat (cf. [4,9,11] and the references therein).

Randers metrics are the simplest non-Riemannian Finsler metrics, which are expressed by $F = \alpha + \beta$ with $\| \beta \|_\alpha < 1$, where α and β are a Riemannian metric and 1-form on an n-dimensional manifold M respectively. It is known that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat, which means that α is of constant sectional curvature by Beltrami's theorem, and β is closed [1]. For instance, the well-known Funk metric

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$$F=\frac{\sqrt{(1-|x|^2)|y|^2+<x,y>^2}}{1-|x|^2}-\frac{<x,y>}{1-|x|^2}$$

is a locally projectively flat Randers metric on the unit ball $\mathbb{B}^n(1)$ with constant flag curvature $K=-\frac{1}{4}$.

Square metrics, given by $F=\frac{(\alpha+\beta)^2}{\alpha}$ with $\|\beta\|_\alpha<1$, are another important kind of non-Riemannian Finsler metrics. The classical example of the square metrics is the Berwald's metric constructed by L. Berwald [2],

$$F=\frac{(\sqrt{(1-|x|^2)|y|^2+<x,y>^2}+<x,y>)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+<x,y>^2}}.$$

Berwald's metric is also projectively flat on $\mathbb{B}^n(1)$ with vanishing flag curvature $K=0$.

In [3], R. Bryant found a locally projectively flat Finsler metric with positive flag curvature $K=1$ on S^2. Later on, in [8], Z. Shen extended Bryant's metric to S^n, which is locally expressed by

$$F=\sqrt{\frac{\sqrt{\mathcal{A}}+\mathcal{B}}{2\mathcal{D}}+\left(\frac{\mathcal{C}}{\mathcal{D}}\right)^2}+\frac{\mathcal{C}}{\mathcal{D}},$$

where

$$\begin{aligned}
\mathcal{A}&:=\big(|y|^2\cos(2\theta)+|x|^2|y|^2-<x,y>^2\big)^2+\big(|y|^2\sin(2\theta)\big)^2;\\
\mathcal{B}&:=|y|^2\cos(2\theta)+|x|^2|y|^2-<x,y>^2;\\
\mathcal{C}&:=<x,y>\sin(2\theta);\qquad \mathcal{D}:=|x|^4+2|x|^2\cos(2\theta)+1,
\end{aligned}$$

where $0<\theta<\pi/2$.

As we know, Randers metrics and square metrics belong to the (α,β)-metrics defined by a Riemannian metric α and a 1-form β on an n-dimensional manifold M,

$$F=\alpha\phi(s),\qquad s=\frac{\beta}{\alpha},$$

where $\phi=\phi(s)>0$ is a smooth function satisfying $\phi-s\phi'+(b^2-s^2)\phi''>0$ such that F is a regular Finsler metric. Z. Shen gave an equivalent characterization for locally projectively flat (α,β)-metrics in [7]. After that, B. Li and Z. Shen classified locally projectively flat (α,β)-metrics with constant flag curvature in [5] and C. Yu gave a complete classification of locally projectively flat (α,β)-metrics when $\dim M\geq 3$ in [12]. However, the Bryant's metric is not an (α,β)-metric. In fact, it belongs to a more general class so called *general* (α,β)*-metrics* expressed by

$$F=\alpha\phi(b^2,s),\quad b:=\|\beta\|_\alpha,\quad s:=\frac{\beta}{\alpha},$$

where $\phi=\phi(b^2,s)$ is a positive smooth function satisfying (2.4) [13]. Obviously, if $\phi_1=0$, then $F=\alpha\phi(b^2,s)$ is exactly an (α,β)-metric, where ϕ_1 is the usual derivative of ϕ with respect to the first variable b^2. We shall denote ϕ_2 by the usual derivative of ϕ with respect to the second variable s. Similar meanings are suitable for ϕ_{11}, ϕ_{12} and ϕ_{22} etc. throughout the paper.

To find more non-trivial locally projectively flat Finsler metrics, B. Li and Z. Shen recently gave an equivalent characterization for locally projectively flat general (α,β)-metrics $F=\alpha\phi(b^2,s)$ on an $n(\geq 3)$-dimensional manifold when α is locally projectively flat and $\phi_1\neq 0$ [6]. Note that $\phi_1\neq 0$ here means that $\phi_1\neq 0$ everywhere for $|s|\leq b<b_0$. Based on this, we obtain a more refined characterization for this class of metrics.