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Parallel and dual surfaces of cuspidal edges

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ABSTRACT

We study parallel surfaces and dual surfaces of cuspidal edges. We give concrete forms of principal curvature and principal direction for cuspidal edges. Moreover, we define ridge points for cuspidal edges by using those. We clarify relations between singularities of parallel and dual surfaces and differential geometric properties of initial cuspidal edges.

1. Introduction

It is well-known that cuspidal edges and swallowtails are generic singularities of wave fronts in $\boldsymbol{R}^3$ (for example, see [1]). There are many studies of wave fronts from the differential geometric viewpoint [3,7–9,13,15]. In particular, various geometric invariants of cuspidal edges were studied by Martins and Saji [8]. To investigate geometric invariants of cuspidal edges, they introduced the normal form of cuspidal edges. On the other hand, parallel surfaces of a regular surface are fronts and might have singularities. Porteous, Fukui and Hasegawa studied the singularities of parallel surfaces and caustics from the viewpoint of singularity theory (cf. [4,11,12]) when the initial surface is regular. Porteous [11,12] introduced the notion of ridge point for regular surfaces relative to principal curvature and principal direction. Using this notion, Fukui and Hasegawa [4] showed relations between singularities of parallel surfaces and geometric properties of initial surfaces.

In this paper, we deal with parallel surfaces when the initial surfaces have singularities. In particular, we consider parallel surfaces of cuspidal edges. Since cuspidal edges have unit normal vector fields, we can con-

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sider parallel surfaces. We show relations between singularities on parallel surfaces and geometric properties of initial cuspidal edges (Theorem 3.2). Ridge points play an important role in studying parallel surfaces of regular surfaces, and also play an important role in investigating this case. Generally, mean curvature is unbounded at cuspidal edges. Thus principal curvatures might be unbounded. We give a condition for one principal curvature to be well-defined (in particular, finite) as a C^∞-function at cuspidal edges (Proposition 2.2). A notion of ridge points for cuspidal edges is defined in Section 2 using principal curvature and principal direction.

In Section 3, we study parallel surfaces of a cuspidal edge from the viewpoint of differential geometry. Moreover, we study the extended distance squared functions on cuspidal edges. In the case of cuspidal edges, the extended distance squared function has D_4 singularities or worse, unlike the case of regular surfaces. We give the conditions for distance squared functions to have D_4 singularities (Theorem 3.3).

In Section 4, we study dual surfaces and the extended height functions. In the case of cuspidal edges, extended height functions have A_2 singularities or worse. We define dual surfaces as a part of the discriminant set of extended height functions. We give relations between singularities of dual surfaces and geometric properties of cuspidal edges (Proposition 4.2) and show conditions for the extended height function to have D_4 singularities. Moreover, we give relations between singularities of dual surfaces and extended height functions (Proposition 4.3).

All maps and functions considered here are of class C^∞ unless otherwise stated.

2. Cuspidal edges

First we recall some properties of wave fronts and frontals. For details, see [1,3,7,9,15].

Let $f : V \to \boldsymbol{R}^3$ be a smooth map and (u, v) be a coordinate system on V, where $V \subset \boldsymbol{R}^2$ is a domain. We call f a *frontal* if there exists a unit vector field ν along f such that $L = (f, \nu) : V \to T_1\boldsymbol{R}^3$ is an isotropic map, where $T_1\boldsymbol{R}^3$ is the unit tangent bundle of $\boldsymbol{R}^3$ equipped with the canonical contact structure, and is identified with $\boldsymbol{R}^3 \times S^2$, where S^2 is the unit sphere. If L gives an immersion, f is called a *wave front* or a *front*. The isotropicity of L is equivalent to the orthogonality condition

$$\langle df(X_p), \nu(p)\rangle, \quad (X_p \in T_pV,\ p \in V).$$

We call ν a *unit normal vector* or the *Gauss map* of f. For a frontal f, the function $\lambda : V \to \boldsymbol{R}$ defined as $\lambda(u, v) = \det(f_u, f_v, \nu)(u, v)$ is called the *signed area density function*, where $f_u = \partial f/\partial u$, $f_v = \partial f/\partial v$. A point $p \in V$ is called a *singular point* of f if f is not an immersion at p. Let $S(f)$ be the set of singular points of f. A singular point $p \in S(f)$ is called *non-degenerate* if $d\lambda(p) \neq 0$ holds. Let $p \in S(f)$ be a non-degenerate singular point. Then, by the implicit function theorem, $S(f)$ is parametrized by a regular curve $\gamma(t) : (-\varepsilon, \varepsilon) \to V$ $(\varepsilon > 0)$ with $\gamma(0) = p$. We call γ a *singular curve* and the direction of $\gamma' = d\gamma/dt$ a *singular direction*. Moreover, there exists a unique non-zero vector field $\eta(t) \in T_{\gamma(t)}V$ up to non-zero functional scalar multiplications such that $df(\eta(t)) = \mathbf{0}$ on $S(f)$. This vector field $\eta(t)$ is called a *null vector field*. A non-degenerate singular point $p \in S(f)$ is said to be of the *first kind* if $\eta(0)$ is transverse to $\gamma'(0)$. Otherwise, it is said to be of the *second kind*.

A *cuspidal edge* is a map-germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at $\mathbf{0}$ and a *swallowtail* is a map-germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at $\mathbf{0}$, where two map-germs $f, g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ are *$\mathcal{A}$-equivalent* if there exist a diffeomorphisms $\Xi_s : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^2, \mathbf{0})$ on the source and $\Xi_t : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ on the target such that $\Xi_t \circ f = g \circ \Xi_s$ holds. The criteria for these singularities are known.

Theorem 2.1. *(See [7, Proposition 1.3].) Let $f : V \to \boldsymbol{R}^3$ be a front and $p \in V$ a non-degenerate singular point of f.*