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Plane curves with curvature depending on distance to a line

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A B S T R A C T

Motivated by a problem posed by David A. Singer in 1999 and by the classical Euler elastic curves, we study the plane curves whose curvature is expressed in terms of the (signed) distance to a line. In this way, we provide new characterizations of some well known curves, like the catenary or the grim-reaper. We also find out several interesting families of plane curves (including closed and embedded ones) whose intrinsic equations are expressed in terms of elementary functions or Jacobi elliptic functions and we are able to get arc length parametrizations of them and they are depicted graphically.

1. Introduction

The fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined up to rigid motion by its curvature given as a function of its arc-length. However, in most cases such curves are impossible to find explicitly in practice, due to the difficulty in solving the quadratures appearing in the integration process. In [9], David A. Singer considered a different sort of problem:

Can a plane curve be determined if its curvature is given in terms of its position?

Having the above comment in mind, it is expectable that if the curvature κ is given by a function of its position, i.e. $\kappa = \kappa(x, y)$, the situation is even more complicated. The general form of this problem translates into solving a nonlinear differential equation:

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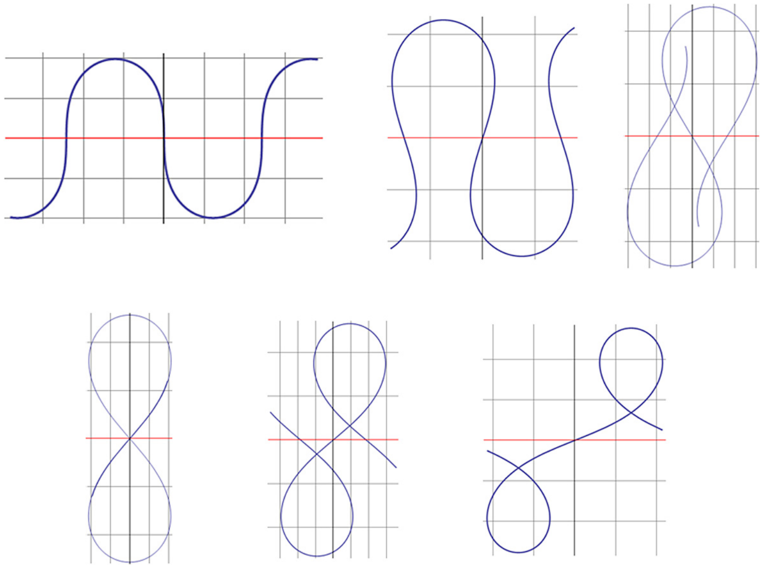

Fig. 1. Euler elastic curves.

$$\frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{3/2}} = \kappa(x(t), y(t))$$

Probably the most interesting solved problem in this setting corresponds to the Euler elastic curves. Classically an elastica is the solution to a variational problem proposed by Daniel Bernoulli to Leonhard Euler in 1744, that of minimizing the bending energy of a thin inextensible wire. From a mathematical point of view, the modelling for this problem consists of minimizing the integral of the squared curvature for curves of a fixed length satisfying given first order boundary data (see [3,10]). These curves satisfy a remarkable property: the curvature is proportional to one of the coordinate functions, say $\kappa(x, y) = 2\lambda y$ (see Section 3). Among the Euler elasticae, there is a unique closed curve with the shape of a figure-eight (see Fig. 1) and the expression of the curvature in terms of the arc parameter involves elliptic Jacobi functions.

Singer in [9] started to study the posed problem by considering the condition $\kappa(x, y) = \sqrt{x^2 + y^2}$ and tried to find analytic representations for these curves. First he proved (see Theorem 3.1 in [9]) that the problem of determining a curve whose curvature is $\kappa(r)$, where r is the distance from the origin, is solvable by quadratures when $r\kappa(r)$ is a continuous function. The proof is based in giving to such curvature an interpretation of a central potential in the plane and finding the trajectories by the standard methods in classical mechanics. See also [11] for a group-theoretical approach of the same result viewing the Frenet equations as a fictitious dynamical system. But the simple case $\kappa(r) = r$, where elliptic integrals appear, illustrated that the fact that the corresponding differential equation is integrable by quadratures does not mean that it is easy to perform the integrations. In [9], only the very pleasant special case of the classical Bernoulli lemniscate, $r^2 = 3\cos 2\theta$ in polar coordinates, was solved explicitly, where the corresponding elliptic integral becomes elementary. Some other families of plane curves whose curvature depends on distance from the origin were considered later. In a chronological order, we emphasize the following: Bernoulli's lemniscate, $\kappa = \sigma\, r$, $\sigma > 0$ (see [9] and [4]); Levy's elasticae, $\kappa = a\, r^2 + c$, $a, c \in \mathbb{R}, a \neq 0$ (see [2]); generalized Sturm spirals, $\kappa = \sigma/r$, $\sigma > 0$ (see [5,7,8]) and deformations of Cassinian ovals: $\kappa = \lambda/r^3 + 3\mu\, r$, $\lambda, \mu \in \mathbb{R}$ (see [6]).

In this article, inspired by Euler elastic curves, we propose to study the plane curves whose curvature depends on distance to a line instead from a point. After a rigid motion, we can suppose this line as the x-axis of the reference system for the curve and so we afford the case $\kappa = \kappa(y)$ of the aforementioned Singer's problem, including the very important family of Euler elastic curves corresponding to the linear function $\kappa(y) = 2\lambda y$.