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Holomorphic Poisson structure and its cohomology on nilmanifolds

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ABSTRACT

The subject for investigation in this note is concerned with holomorphic Poisson structures on nilmanifolds with abelian complex structures. As a basic fact, we establish that on such manifolds, the Dolbeault cohomology with coefficients in holomorphic polyvector fields is isomorphic to the cohomology of invariant forms with coefficients in invariant polyvector fields.

We then quickly identify the existence of invariant holomorphic Poisson structures. More important, the spectral sequence of the Poisson bi-complex associated to such holomorphic Poisson structure degenerates at E_2. We will also provide examples of holomorphic Poisson structures on such manifolds so that the related spectral sequence does not degenerate at E_2.

1. Introduction

The investigation of Poisson bracket from a complex perspective started a while ago [24]. Attention on this subject in the past ten years is largely due to its role in generalized complex geometry [13–15]. It is now also known that product of holomorphic Poisson structures together with symplectic structures forms the local model of all generalized complex geometry [2].

Therefore, one could take many different routes when investigating holomorphic Poisson structure. One could study it as a complex geometric object and study its deformations as in [16]. One could also investigate it in the context of extended deformation, or generalized complex structures [11,25]. In the former case, the deformation theory is dictated by the differential Gerstenhaber structure on a cohomology ring with coefficients in the holomorphic polyvector fields. In the latter case, it is on the part of the cohomology with total degree-2 as explained in [25]. As a key feature of generalized complex geometry is to encompass classical complex structures with symplectic structures in a single geometric framework, one could also

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relate the cohomology theory of a holomorphic Poisson structure as a generalized complex manifold to the cohomology theory on symplectic geometry [1,28].

In this paper, we consider holomorphic Poisson structure as a geometric object in generalized complex structure, and study its cohomology theory accordingly. It will set a stage for studying deformation theory. In this perspective, it is known that the cohomology of a holomorphic Poisson structure could be computed by a bi-complex [17,18]. The first level of the associated spectral sequence of this bi-complex is the Dolbeault cohomology with coefficients in holomorphic polyvector fields. It is known that this spectral sequence often, but not always degenerates at its second level [5]. It is therefore interesting to find how often this spectral sequence indeed degenerates at its second level. On Kählerian manifolds, an affirmative answer for complex surfaces is found, and other general observations are made in [5].

In this note, we focus on non-Kählerian manifolds. In particular, we focus on nilmanifolds due to their rich history and role in generalized complex geometry [4]. Investigation on the cohomology theory on nilmanifolds also has a very rich history, beginning with Nomizu's work on de Rham cohomology [23]. There has been a rich body of work on the Dolbeault cohomology of nilmanifolds with invariant complex structures [7,9], and work on Dolbeault cohomology on the same kind of manifolds with coefficients in holomorphic tangent bundle [6,8,12,21,26]. In favorable situations, various authors proved that the cohomology is isomorphic to the cohomology of invariant objects.

In this paper, after a review of holomorphic Poisson structures and their associated bi-complex structures and a brief review of abelian complex structures on nilmanifolds, we show that the Dolbeault cohomology of an abelian complex structure on a nilmanifold with coefficients in holomorphic polyvector fields is isomorphic to the cohomology of the corresponding invariant objects; see Theorem 1. It means that the cohomology could be computed by a differential algebra over the field of complex numbers. It enables an analysis of the spectral sequence of the bi-complex associated to an invariant holomorphic Poisson structure.

After we establish the existence of invariant holomorphic Poisson structures on nilmanifolds with abelian complex structures in Section 5, we focus on proving Theorem 2. This theorem, which is also the key observation in this paper, states that on any nilmanifold with abelian complex structures, there exists an invariant holomorphic Poisson structure such that the spectral sequence of its associated bi-complex degenerates at its second level. This result generalizes one of the observations in [5] where the authors could only work on 2-step nilmanifolds.

However, at the end of this note, we caution the readers with an example that although such degeneracy occurs often, but it is not always true even in the context of nilmanifolds with abelian complex structures.

2. Holomorphic Poisson cohomology

In this section, we review the basic background materials as seen in [5] to set up the notations.

Let M be a manifold with an integrable complex structure J. Its complexified tangent bundle $TM_{\mathbb{C}}$ splits into the direct sum of bundle of $(1,0)$-vectors $TM^{1,0}$ and bundle of $(0,1)$-vectors $TM^{0,1}$. Their p-th exterior products are respectively denoted by $TM^{p,0}$ and $TM^{0,p}$. Denote their dual bundles by $TM^{*(p,0)}$ and $TM^{*(0,p)}$ respectively.

When X, Y are vector fields, we denote their Lie bracket by $[X,Y]$. When ω is a 1-form, we denote the Lie derivative of ω along X by $[X,\omega]$. When ρ is another 1-form, we set $[\omega,\rho]=0$. With this "bracket" structure and the natural projection from $\overline{L} := TM^{0,1} \oplus TM^{*(1,0)}$ to the summand $TM^{0,1}$, the bundle $\overline{L}$ is equipped with a complex Lie algebroid structure. Together with its conjugate bundle $L = TM^{1,0} \oplus TM^{*(0,1)}$, they form a Lie bi-algebroid [19]. Then we get the Lie algebroid differential $\overline{\partial}$ for the Lie algebroid $\overline{L}$ [20].

$$\overline{\partial} : C^{\infty}(M, TM^{1,0} \oplus TM^{*(0,1)}) \to C^{\infty}(M, \wedge^2(TM^{1,0} \oplus TM^{*(0,1)})). \tag{1}$$