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Stability of stationary maps of a functional related to pullbacks of metrics

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ABSTRACT

Let (M, g) and (N, h) be Riemannian manifolds without boundary. We consider the functional

$$\Phi(f) = \int_M \|f^*h\|^2 \, \mathrm{d}v_g$$

for any smooth map $f : M \to N$, where $\mathrm{d}v_g$ is the volume form on (M, g), and $\|f^*h\|$ denotes the norm of the pullback f^*h of the metric h by the map f. We study stationary maps for the functional $\Phi(f)$, and show that stable stationary maps from or into minimal submanifolds in the unit spheres are rare if Ricci curvatures of submanifolds are large. Symmetric spaces of some type, which are minimally and isometrically immersed in the unit spheres, are treated in detail.

1. Introduction and statements of results

Let (M, g) and (N, h) be Riemannian manifolds without boundary, and let f be a smooth map from M to N. In this paper we are concerned with a functional related to the pullback f^*h of the metric h by the map f, i.e.,

$$(f^*h)(X, Y) = h\big(\mathrm{d}f(X), \mathrm{d}f(Y)\big)$$

for any vector fields X, Y on M. In [8] and [9], the second author introduced a functional

$$\Phi(f) = \int_M \|f^*h\|^2 \, \mathrm{d}v_g$$

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where $\mathrm{d}v_g$ is the volume form on (M, g), and $\|f^*h\|$ denotes the norm of the 2-tensor f^*h. Thus we have

$$\|f^*h\|^2 = \sum_{i,\,j} h\big(\mathrm{d}f(e_i), \mathrm{d}f(e_j)\big)^2$$

where $\{e_i\}$ is a local orthonormal frame on (M, g).

The energy density $\|\mathrm{d}f\|^2 = \sum_i h(\mathrm{d}f(e_i), \mathrm{d}f(e_i))$ of f which appears in the theory of harmonic maps is considered as $\mathrm{trace}_g(f^*h)$. In our functional Φ, the density is replaced by the squared norm of f^*h. In their paper [2], J. Eells and H. Sampson suggested to make densities of functionals from symmetric functions of eigenvalues of f^*h with respect to g (p. 113). For example the first and the n-th (n is the dimension of the manifold) fundamental symmetric functions produce the energy density and the volume density respectively.

Our density is one of this type. The functional Φ is not only alike the energy functional in its appearance, it has quite similar geometric properties as the energy functional. Indeed in [3], the authors investigated the stable stationary maps (i.e., with nonnegative second variation) from the spheres or into the spheres and showed that they are constant in high dimensional cases. Related results are well known and important in the theory of harmonic maps.

In this paper, we work on immersed minimal submanifold of spheres, and show that stable stationary maps from or into immersed minimal submanifold of spheres are quite rare if the Ricci curvature of the minimal submanifold is sufficiently large. These results are generalizations of those in [3] and also the counterpart of the investigations in [12,6,11] and [14] for harmonic maps.

From now on, the immersed image in the sphere (or in the ambient Euclidean space) and the source manifold will not be distinguished. For simplicity, we call stationary maps of the functional Φ *symphonic* maps following the terminology in [5] and [10]. The main results are the following.

Theorem 1. *Let (M, g) be an m-dimensional compact connected immersed minimal submanifold of the standard unit sphere. Suppose that the Ricci curvature Ric_g of g satisfies $\mathrm{Ric}_g > \frac{3}{4}mg$. Then any stable symphonic map f from M into any Riemannian manifold is a constant map.*

The assumption on the Ricci curvature is sharp. See Remark 2 in Section 3 for details. We also have the following result dual to Theorem 1.

Theorem 2. *Let (N, h) be an n-dimensional immersed minimal submanifold of the standard unit sphere. Suppose that the Ricci curvature Ric_h of h satisfies $\mathrm{Ric}_h > \frac{3}{4}nh$. Then any stable symphonic map f from any compact connected Riemannian manifold into N is a constant map.*

The assumption on the Ricci curvature is also sharp. See Remark 3 in Section 4. On an Einstein manifold, the stability of the identity map and the first eigenvalue of the Laplacian acting on functions have nice relations which will be used to investigate the case of symmetric spaces.

Theorem 3. *Let (M, g) be an n-dimensional compact connected Einstein manifold with $\mathrm{Ric}_g = \rho_0 g$, where ρ_0 is a constant. Then the following two conditions are equivalent:*

(1) *The first eigenvalue λ_1 of the Laplacian Δ acting on functions satisfies $\lambda_1 < \frac{4}{3}\rho_0$.*
(2) *The identity map of M is an unstable symphonic map.*

A simply-connected irreducible Riemannian symmetric space (M, g) of compact type (always assumed to be connected.) is an Einstein manifold. By Takahashi's theorem [13], (M, g) can be isometrically and minimally immersed in a (not necessarily unit) Euclidean sphere. Hence we obtain the following result.