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Hopf hypersurfaces in pseudo-Riemannian complex and para-complex space forms

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ABSTRACT

The study of real hypersurfaces in pseudo-Riemannian complex space forms and para-complex space forms, which are the pseudo-Riemannian generalizations of the complex space forms, is addressed. It is proved that there are no umbilic hypersurfaces, nor real hypersurfaces with parallel shape operator in such spaces. Denoting by J be the complex or para-complex structure of a pseudo-complex or para-complex space form respectively, a non-degenerate hypersurface of such space with unit normal vector field N is said to be *Hopf* if the tangent vector field JN is a principal direction. It is proved that if a hypersurface is Hopf, then the corresponding principal curvature (the *Hopf* curvature) is constant. It is also observed that in some cases a Hopf hypersurface must be, locally, a tube over a complex (or para-complex) submanifold, thus generalizing previous results of Cecil, Ryan and Montiel.

0. Introduction

The study of real hypersurfaces in complex space forms, i.e. the complex projective space $\mathbb{CP}^n$ and the complex hyperbolic space $\mathbb{CH}^n$, has attracted a lot of attention in the last decades (see [10] for a survey of the subject and references therein). The complex structure J of a complex space form induces a rich structure on real hypersurface; in particular, on an arbitrary oriented hypersurface $\mathcal{S}$ of $\mathbb{CP}^n$ or $\mathbb{CH}^n$ with unit vector normal field N, a canonical tangent field, called *the structure vector field* or *the Reeb vector field*, is defined by $\xi := -JN$. If ξ is a principal direction on $\mathcal{S}$, i.e. an eigenvector of the shape operator, $\mathcal{S}$ is called a *Hopf hypersurface*. It turns out that the principal curvature associated with the structure vector ξ (the *Hopf principal curvature*) of a connected, Hopf hypersurface must be constant (this was proved in [8] in the projective case and in [6] in the hyperbolic case). Moreover, in [2], Hopf hypersurfaces in $\mathbb{CP}^n$ are locally characterized as tubes over complex submanifolds, while in [9], the same statement is proved for

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Hopf hypersurfaces of $\mathbb{CH}^n$ whose Hopf principal curvature a satisfies $|a| > 2$. Recently Hopf hypersurfaces of $\mathbb{CH}^n$ with small Hopf principal curvature, i.e. satisfying $|a| \leq 2$, have been studied through a kind of generalized Gauss map in [4] and [5], while in [7] a unified approach is proposed, relating Hopf hypersurfaces to totally complex (or para-complex) submanifolds of some natural quaternionic manifold.

The purpose of this paper is to address the study of real hypersurfaces in *pseudo-complex space forms* $\mathbb{CP}^n_p$, which are the pseudo-Riemannian generalizations of the complex space forms, and in *para-complex space form* $\mathbb{DP}^n$. The latter space is the para-complex analog of $\mathbb{CP}^n$ and is equipped with both a pseudo-Riemannian metric and a *para-complex* structure, still denoted by J, which satisfies $J^2 = Id$. Furthermore, given a real hypersurface in $\mathbb{DP}^n$ with non-degenerate induced metric, the Hopf field is defined exactly as in the complex case. We refer to the next section for the precise definition of $\mathbb{DP}^n$ and a brief description of its geometry. Since both the pseudo-complex and the para-complex case will be studied simultaneously, we define ϵ in such way that $J^2 = -\epsilon Id$, i.e. $\epsilon = 1$ corresponds to the complex case and $\epsilon = -1$ to the para-complex case. Moreover, $\mathcal{M}$ will denote the pseudo-Riemannian complex space form $\mathbb{CP}^n_p$ or the para-complex space form $\mathbb{DP}^n$, with holomorphic or para-holomorphic curvature $4c$, where $c := \pm 1$.

Our results are:

Theorem 1. *There exist no umbilic real hypersurface, nor real hypersurface with parallel shape operator, in $\mathcal{M}$.*

Theorem 2. *Let $\mathcal{S}$ be a connected, non-degenerate hypersurface of $\mathcal{M}$ which is Hopf, i.e. its structure vector ξ is a principal direction of $\mathcal{S}$. Then the corresponding principal curvature a, i.e. defined by $A\xi = a\xi$, is constant.*

Theorem 3. *Let $\mathcal{S}$ be a connected, non-degenerate hypersurface of $\mathcal{M}$ with unit normal N. Assume that $\mathcal{S}$ is Hopf and denote by a the corresponding principal curvature, i.e. $A\xi = a\xi$. Then if $c\,\epsilon\langle N, N\rangle = 1$, or if $c\,\epsilon\langle N, N\rangle = -1$ and $|a| > 2$, then $\mathcal{S}$ is, locally, a tube over a complex or para-complex submanifold.*

Remark 1. In the case $c = 1$, $\epsilon = 1$ and $p = 0$, $\mathcal{M}$ is the complex projective space $\mathbb{CP}^n$, and if $c = -1$, $\epsilon = 1$ and $p = n$, we have $\mathcal{M} = \mathbb{CH}^n$, the complex hyperbolic space. Hence Theorem 3 generalizes [2] and [9]. Observe that in these two cases, the metric being positive, we have $\langle N, N\rangle = 1$.

This paper is organized as follows: in Section 1 the geometry of the pseudo-Riemannian complex and the para-complex space forms is described. Section 2 contains basic relations about the geometry of real hypersurfaces in $\mathcal{M}$ and the proof of Theorem 1. In Section 3 four lemmas about real hypersurfaces and the proof of Theorem 2 are presented. Finally, in Section 4 the proof of Theorem 3 is given and at the end of the section some open problems are proposed for further research on this area.

1. The ambient spaces: pseudo-Riemannian complex and para-complex space forms

1.1. The abstract structures

All along the paper the ambient space will be a $2n$-dimensional pseudo-Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle, J)$ endowed with a complex or para-complex structure J, i.e. a $(1, 1)$ tensor field satisfying $J^2 = -\epsilon Id$ which is compatible with respect to $\langle\cdot,\cdot\rangle$, i.e.

$$\langle J\cdot, J\cdot\rangle = \epsilon\langle\cdot,\cdot\rangle.$$