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Curvature conditions for complex-valued harmonic morphisms

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ABSTRACT

We study the curvature of manifolds which admit a complex-valued submersive harmonic morphism with either, totally geodesic fibers or that is holomorphic with respect to a complex structure which is compatible with the second fundamental form.

We also give a necessary curvature condition for the existence of complex-valued harmonic morphisms with totally geodesic fibers on Einstein manifolds.

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1. Introduction

A harmonic morphism is a map between two Riemannian manifolds that pulls back local harmonic functions to local harmonic functions. The simplest examples of harmonic morphisms are constant maps, real-valued harmonic functions and isometries. A characterization of harmonic morphisms was given by Fuglede and Ishihara, they showed in [1] and [2], respectively, that harmonic morphisms are exactly the horizontally weakly conformal harmonic maps. If we restrict our attention to the maps where the codomain is a surface then the harmonic morphisms are the horizontally weakly conformal maps with minimal fibers at regular points by a result of Baird and Eells [3].

Between two surfaces the harmonic morphisms are exactly the weakly conformal maps. Since the composition of two harmonic morphisms is again a harmonic morphism, we get that, locally any harmonic morphism to a surface can be turned into a harmonic morphism to the complex plane by composing with a weakly conformal map.

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Local existence of harmonic morphisms can be characterized in terms of foliations. If the codomain is a surface then the existence of a local harmonic morphism is equivalent to the existence of a local conformal foliation with minimal fibers at regular points, see [4] by Wood.

Baird and Wood found a necessary condition, see [5] Corollary 4.4, on the curvature for local existence of complex-valued harmonic morphisms on three-manifolds. In this case the fibers are geodesics and there is an orthonormal basis $\{X, Y\}$ for the horizontal space such that the **Ricci curvature condition**

$$\mathrm{Ric}(X, X) = \mathrm{Ric}(Y, Y) \text{ and } \mathrm{Ric}(X, Y) = 0,$$

is satisfied. In three dimensions this is equivalent to

$$\langle R(X, U)U, X\rangle = \langle R(Y, U)U, Y\rangle \text{ and } \langle R(X, U)U, Y\rangle = 0$$

for any vertical unit vector U, which in turn is equivalent to the fact that the sectional curvature $K(X_\theta \wedge U)$ is independent of θ where $X_\theta = \cos(\theta)X + \sin(\theta)Y$.

We show in this paper that the last condition is true for any complex-valued submersive harmonic morphism with totally geodesic fibers.

Theorem 1.1. *Let (M, g) and (N^2, h) be Riemannian manifolds, let $\phi : (M, g) \to (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers and $p \in M$. Given any $U, V \in \mathcal{V}_p = \ker(\mathrm{d}\,\phi)$ and any orthonormal basis $\{X, Y\}$ for $\mathcal{H}_p = \mathcal{V}_p^\perp$, set $X_\theta = \cos(\theta)X + \sin(\theta)Y$. Then*

$$\langle R(X_\theta \wedge U), X_\theta \wedge V\rangle$$

is independent of θ.

In four dimensions or more this is stronger than the Ricci curvature condition. Note that Examples 6.1 and 6.2 of [6] by Gudmundsson and Svensson do not have totally geodesic fibers. So they are counterexamples to the Ricci curvature condition only in the case of minimal but not totally geodesic fibers.

If we assume that the domain (M, g) is an Einstein manifold, then the curvature operator, in a suitably chosen basis, splits into two blocks and we find that there are at least $\dim(M) - 2$ double eigenvalues for the curvature operator. We use this to give an example of a five dimensional homogeneous Einstein manifold that does not have any submersive harmonic morphism with totally geodesic fibers.

Harmonic morphisms with totally geodesic fibers have been studied in different ways before. Baird and Wood, Section 6.8 [7], classify them in the constant curvature case. Later Pantilie generalized to the case where the domain is conformaly equivalent to constant curvature [8]. Mustafa [9] gave a Bochner type curvature formula and applied it to foliations with large codimension.

We end this paper by showing that the Ricci curvature condition is satisfied by harmonic morphisms that are holomorphic with respect to a complex structure and where the second fundamental form is compatible with the complex structure. The result is similar to Proposition 6.3 from [10], where Loubeau and Pantilie describe twistorial harmonic morphisms, but only in 4 dimensions.

2. The curvature condition

Let (M, g) and (N, h) be Riemannian manifolds and let $\phi : (M, g) \to (N, h)$ be a smooth submersion. Denote the vertical distribution associated with ϕ by $\mathcal{V} = \ker(\mathrm{d}\,\phi)$ and the horizontal distribution by $\mathcal{H} = \mathcal{V}^\perp$. For two vector fields E, F on M define the tensors A and B, introduced in [11], by

$$A_E F = \mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) \text{ and } B_E F = \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F).$$

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