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Lower volume growth and total σ_k-scalar curvature estimates

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Let M be a complete noncompact immersed submanifold in a Hadamard manifold $\bar{M}$ and Φ a positive-semidefinite symmetric endomorphism on M. Under our assumptions, we obtain that either $\Phi \equiv 0$ or the growth of the integral $\int_{B_\mu} \mathrm{trace}(\Phi)\, d\mathrm{Vol}$ is at least logarithmic. As the main application, we given conditions to guarantee that the total σ_k-scalar curvature is infinite. Further applications to convex functions and certain ruled manifolds are given.

1. Introduction

Let $f : M^m \to \bar{M}$ be an isometric immersion of an m-dimensional Riemannian manifold M in a Riemannian manifold $\bar{M}$ and II be the second fundamental form of f. Let $\mathcal{S}(TM)$ be the C^1 bundle of the symmetric endomorphism on M and $\mathcal{S}^+(TM)$ be the subset of positive-semidefinite endomorphisms of $\mathcal{S}(TM)$. In 2002, J. Grosjean [11] introduced the following object associated with the immersion f.

Definition 1.1. The *mean curvature vector field of the immersion f with respect to an endomorphism* $\Phi \in \mathcal{S}(TM)$ is the normal vector field $H_\Phi : M \to T^\perp M$ defined by the trace:

$$H_\Phi = \mathrm{trace}\left\{(X, Y) \in TM \times TM \mapsto II(\Phi(X), Y)\right\}.$$

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The object above reproduces naturally some important objects in the immersion theory. More precisely, it generalizes the mean curvature vector H and, in codimension one, it also includes the higher order mean curvatures of the immersion f, by considering the endomorphism Φ as the Newton's transformations associated with the shape operator of the hypersurface. So, the concept of H_Φ is very natural.

In the next definition, we recall the important concept of the divergence of a tensor, originally defined in the tensorial calculus.

Definition 1.2. The *divergence of* Φ is the tangent vector field on M defined by

$$\operatorname{div}(\Phi) = \mathit{trace}(\nabla\Phi).$$

We observe that if $\Phi = \lambda I$, where λ is a C^1 function on M and $I \in \mathcal{S}(TM)$ is the identity endomorphism, then it is easy to show that $\operatorname{div}(\Phi)$ coincides with the gradient vector field $\nabla\lambda$.

It is a well-known fact that a complete noncompact minimal submanifold in a Hadamard manifold must have infinite volume (see, for instance, Theorem 1 in Appendix of [9] or Lemma 1 of [5]). Our main theorem says the following:

Theorem 1.1. *Let $f : M \to \bar{M}$ be an isometric immersion of a complete noncompact manifold M in a complete simply-connected manifold $\bar{M}$ with nonpositive radial curvature with respect to some basis point $q_0 \in f(M)$. Let $\Phi \in \mathcal{S}^+(TM)$. Assume that $\Phi(q_0) \not\equiv 0$ and, for some $\bar{\mu}_0 > 0$, it holds*

$$\int_{B_\mu(q_0)} |H_\Phi + \operatorname{div}(\Phi)|\, r \,\mathrm{dVol} \le \int_{B_\mu(q_0)} \beta(r)\,\operatorname{trace}(\Phi)\,\mathrm{dVol}, \tag{1}$$

for all $\mu \ge \bar{\mu}_0$, where $r = d_{\bar{M}}(\cdot\,, q_0)$ is the distance in $\bar{M}$ from q_0 and β is any nondecreasing C^1 function on $[0,\infty)$ with $0 \le \beta(0) < 1$ and $\beta \le 1$. Then, for all $\mu_0 > \bar{\mu}_0$, there exist $c > 0$, depending only on q_0, μ_0 and M, such that

$$\int_{B_\mu(q_0)} \operatorname{trace}(\Phi)\,\mathrm{dVol} \ge c \int_{\mu_0}^{\mu} e^{-\int_{\mu_0}^{\tau} \frac{\beta(s)}{s} ds} d\tau,$$

for all $\mu \ge \mu_0$. In particular, the growth of the integrals $\int_{B_\mu(q_0)} \operatorname{trace}(\Phi)\,\mathrm{dVol}$, with $\mu > \mu_0$, is at least logarithm.

As a direct consequence of Theorem 1.1, by taking $\beta = 0$ in (1), we obtain

Corollary 1.1. *Let $f : M \to \bar{M}$ be an isometric immersion of a complete noncompact manifold M in a Hadamard manifold $\bar{M}$. Let $\Phi \in \mathcal{S}^+(TM)$. Assume that $\operatorname{div}\Phi = H_\Phi = 0$. Then, either $\Phi = 0$ or, for all $\mu_0 > 0$ and $q \in M$ with $\Phi(q) \not\equiv 0$, there exists $c = c(q, \mu_0, M) > 0$ such that*

$$\int_{B_\mu(q)} \operatorname{trace}(\Phi)\,\mathrm{dVol} \ge c\,(\mu - \mu_0),$$

for all $\mu \ge \mu_0$.

Now, we will see different applications for Theorem 1.1 and Corollary 1.1. The first one is a non-direct application of Theorem 1.1 that will be proved in Section 3 below. Note that ruled surfaces in $\mathbb{R}^n$ are examples among the next result applies.