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Construction of lattices of solvable Lie groups from a viewpoint of matrices

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ABSTRACT

In this paper, we reconstruct lattices of famous solvable Lie groups from a viewpoint of matrices. As an application, we construct examples of solvable Lie groups which admit lattices.

0. Introduction

In this paper, we consider lattices of solvable Lie groups. It is well-known that a compact homogeneous space of a connected nilpotent Lie group, which is called a compact nilmanifold, can be expressed of the form $\Gamma\backslash N$, where N is a simply-connected nilpotent Lie group and Γ is a lattice of N. Nomizu [7] has proven that the de Rham cohomology groups of a compact nilmanifold $\Gamma\backslash N$ are isomorphic to the cohomology groups of the Lie algebra $\mathfrak{n}$ of N. Sakane [8] has proven that the Dolbeault cohomology groups of a compact complex parallelizable nilmanifold $\Gamma\backslash N$ can be computed by using the cohomology groups of the Lie algebra $\mathfrak{n}$ of N. Thus, the de Rham cohomology groups of a compact nilmanifold and the Dolbeault cohomology groups of a compact complex parallelizable nilmanifold are independent of lattices.

On the other hand, de Rham cohomology groups and Dolbeault cohomology groups of a solvmanifold of the form $\Gamma\backslash G$, where G is a simply-connected solvable Lie group and Γ is a lattice of G, are dependent on lattices in general [6]. Inoue surfaces, which are contained in class VII of the Enriqueas–Kodaira classification, are also solvmanifolds (cf. [5]), where a solvmanifold is a homogeneous space of solvable Lie group. Hence, it

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is important to construct a lot of lattices of a solvable Lie group and examples of solvable Lie groups which admit lattices concretely. Note that there exist compact solvmanifolds that are not quotients by lattices, for example, the Klein bottle. However, every solvmanifold is finitely covered by a solvmanifold of the form $\Gamma\backslash G$.

In this paper, we construct lattices of solvable Lie groups from a viewpoint of matrices, some of which have already been known (e.g. Examples 3.1, 3.2, 4.3), however, we get a systematic construction of lattices and new lattices of solvable Lie groups in [9]. We also get new solvmanifolds, which can be considered as generalizations of Inoue surfaces of type S^0 (Proposition 4.1, Examples 4.4, 4.5).

1. Construction of 2-step nilpotent Lie groups

In this section, we construct 2-step nilpotent Lie groups and isomorphisms by using bilinear forms (cf. [2, Chapter I-1]).

Let V^n be an n dimensional real vector space, and $B : V \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}^m)$ a linear mapping. We define a multiplication on the set $N(B) = V \times \mathbb{R}^m$ by

$$(v_1, z_1) \cdot (v_2, z_2) = (v_1 + v_2, z_1 + z_2 + (B(v_1))(v_2)),\ v_i \in V, z_i \in \mathbb{R}^m\ (i = 1, 2).$$

It is straightforward to verify that $(v, z)^{-1} = (-v, -z + (B(v))(v))$. Since $Z = 0 \times \mathbb{R}^m$ is a normal subgroup and $N(B)/Z \cong V$ is abelian, $N(B)$ is a 2-step nilpotent Lie group. If $v = {}^t(x_1, \ldots, x_n) = \mathbf{x}$ and $B(\mathbf{x}) = (a_{ij}(\mathbf{x}))$, $i = 1, \ldots, m, j = 1, \ldots, n$ relative to this basis of V, then

$$(\mathbf{x}, t) \mapsto \begin{pmatrix} I_m & B(\mathbf{x}) & z \\ 0 & I_n & \mathbf{x} \\ 0 & 0 & 1 \end{pmatrix},$$

where I_n is the $n \times n$ unit matrix, is a faithful representation of $N(B)$. Let $m = n$. Then we can write $B(\mathbf{x}) = A(\mathbf{x}) + S(\mathbf{x})$, where $A(\mathbf{x})$ is the alternate matrix and $S(\mathbf{x})$ is the symmetric matrix corresponding to $B(\mathbf{x})$, respectively. We can consider a 2-step nilpotent Lie group $N(A)$. Let a subscript 0 denote that the element is in $N(A)$ and a subscript 1 denote that the element is in $N(B) = N(A + S)$. Let

$$\pi_S((\mathbf{x}, z)_0) = (\mathbf{x}, z + \frac{1}{2}(S(\mathbf{x}))(\mathbf{x}))_1.$$

If $(S(\mathbf{x}_1))(\mathbf{x}_2) = (S(\mathbf{x}_2))(\mathbf{x}_1)$ for each $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^n$, then π_S is an isomorphism.

In the case of $m = 1$, B can be considered as a bilinear form. Hence, we write $(B(v_1))(v_2) = B(v_1, v_2)$. If $v = {}^t(x_1, \ldots, x_n) = \mathbf{x}$ and $B = (a_{ij})$, $i, j = 1, \ldots, n$ relative to this basis of V, then we can write the above representation as

$$(\mathbf{x}, z) \mapsto \begin{pmatrix} 1 & {}^t\mathbf{x}B & z \\ 0 & I_n & \mathbf{x} \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $B = A+S$, where A is the alternating bilinear form and S is the symmetric bilinear form corresponding to $B(\mathbf{x})$, respectively. Then we see that π_S is always an isomorphism (see [2]).

Example 1.1. Let us consider the 3-dimensional Heisenberg Lie group

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$