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On equivariant homeomorphisms of boundaries of CAT(0) groups and Coxeter groups ☆

Tetsuya Hosaka

Department of Mathematics, Shizuoka University, Suruga-ku, Shizuoka 422-8529, Japan

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A B S T R A C T

In this paper, we investigate an equivariant homeomorphism of the boundaries ∂X and ∂Y of two proper CAT(0) spaces X and Y on which a CAT(0) group G acts geometrically. We provide a sufficient condition and an equivalent condition to obtain a G-equivariant homeomorphism of the boundaries ∂X and ∂Y as a continuous extension of the quasi-isometry $\phi : Gx_0 \to Gy_0$ defined by $\phi(gx_0) = gy_0$, where $x_0 \in X$ and $y_0 \in Y$. In this paper, we say that a CAT(0) group G is *equivariant (boundary) rigid*, if G determines its ideal boundary by the equivariant homeomorphisms as above. As an application, we introduce some examples of (non-)equivariant rigid CAT(0) groups and we show that if Coxeter groups W_1 and W_2 are equivariant rigid as reflection groups, then so is $W_1 * W_2$. We also provide a conjecture on non-rigidity of boundaries of some CAT(0) groups.

1. Introduction

In this paper, we investigate an equivariant homeomorphism of the boundaries of two proper CAT(0) spaces on which a CAT(0) group acts geometrically as a continuous extension of a quasi-isometry of the two CAT(0) spaces.

Definitions and details of CAT(0) spaces and their boundaries are found in [6] and [21]. A *geometric* action on a CAT(0) space is an action by isometries which is proper [6, p.131] and cocompact. We note that every CAT(0) space on which some group acts geometrically is a proper space [6, p.132]. A group G is called a *CAT(0) group*, if G acts geometrically on some CAT(0) space X.

It is well-known that if a Gromov hyperbolic group G acts geometrically on a negatively curved space X, then the natural map $G \to X$ $(g \mapsto gx_0)$ extends continuously to an equivariant homeomorphism of the

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E-mail address: hosaka.tetsuya@shizuoka.ac.jp.

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boundaries of G and X. Also if a Gromov hyperbolic group G acts geometrically on negatively curved spaces X and Y, then the boundaries of X and Y are G-equivariant homeomorphic. Indeed the natural map $Gx_0 \to Gy_0$ ($gx_0 \mapsto gy_0$) extends continuously to a G-equivariant homeomorphism of the boundaries of X and Y. The boundaries of Gromov hyperbolic groups are quasi-isometric invariant (cf. [6,10,21–23]).

Here in [23], Gromov asked whether the boundaries of two CAT(0) spaces X and Y are G-equivariant homeomorphic whenever a CAT(0) group G acts geometrically on the two CAT(0) spaces X and Y. In [5], P.L. Bowers and K. Ruane have constructed an example that the natural quasi-isometry $Gx_0 \to Gy_0$ ($gx_0 \mapsto gy_0$) does not extend continuously to any map between the boundaries ∂X and ∂Y of X and Y. Also S. Yamagata [49] has constructed a similar example using a right-angled Coxeter group and its Davis complex. Moreover, there is a research by C. Croke and B. Kleiner [12] on an equivariant homeomorphism of the boundaries ∂X and ∂Y.

Also, C. Croke and B. Kleiner [11] have constructed a CAT(0) group G which acts geometrically on two CAT(0) spaces X and Y whose boundaries are not homeomorphic, and J. Wilson [48] has proved that this CAT(0) group has uncountably many boundaries. Recently, C. Mooney [40] has showed that the knot group G of any connected sum of two non-trivial torus knots has uncountably many CAT(0) boundaries.

Also, it has been observed by M. Bestvina [3] that all the boundaries of a given CAT(0) group are shape equivalent and R. Geoghegan and P. Ontaneda have proved this in [20]. Bestvina has asked the question whether all the boundaries of a given CAT(0) group are cell-like equivalent. This question is an open problem and there are some resent research (cf. [41]).

The purpose of this paper is to provide a sufficient condition and an equivalent condition to obtain a G-equivariant homeomorphism between the two boundaries ∂X and ∂Y of two CAT(0) spaces X and Y on which a CAT(0) group G acts geometrically as a continuous extension of the natural quasi-isometry $Gx_0 \to Gy_0$ ($gx_0 \mapsto gy_0$), where $x_0 \in X$ and $y_0 \in Y$.

Now we recall the example of Bowers and Ruane in [5]. Let $G = F_2 \times \mathbb{Z}$ and $X = Y = T \times \mathbb{R}$, where F_2 is the rank 2 free group generated by $\{a, b\}$ and T is the Cayley graph of F_2 with respect to the generating set $\{a, b\}$. Then we define the action "$\cdot$" of the group G on the CAT(0) space X by

$$(a,0)\cdot(t,r) = (a\cdot t, r),$$
$$(b,0)\cdot(t,r) = (b\cdot t, r),$$
$$(1,1)\cdot(t,r) = (t, r+1),$$

for each $(t,r) \in T \times \mathbb{R} = X$, and also define the action "$*$" of the group G on the CAT(0) space Y by

$$(a,0) * (t,r) = (a\cdot t, r),$$
$$(b,0) * (t,r) = (b\cdot t, r+2),$$
$$(1,1) * (t,r) = (t, r+1),$$

for each $(t,r) \in T \times \mathbb{R} = Y$. Then the group G acts geometrically on the two CAT(0) spaces X and Y, and the quasi-isometry $g \cdot x_0 \mapsto g * y_0$ (where $x_0 = (1,0) \in X$ and $y_0 = (1,0) \in Y$) does not extend continuously to any map from ∂X to ∂Y. Indeed for $g_i = a^i b^i \in F_2$, $\{g_i^\infty \,|\, i \in \mathbb{N}\} \to a^\infty$ as $i \to \infty$ in ∂T,

$$\lim_{n\to\infty} (g_i^n, 0) \cdot x_0 = [g_i^\infty, 0],$$
$$\lim_{n\to\infty} (a^n, 0) \cdot x_0 = [a^\infty, 0],$$

in $X \cup \partial X$, and