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A simple proof of an isoperimetric inequality for Euclidean and hyperbolic cone-surfaces

Ivan Izmestiev [*,1]

Department of Mathematics and Computer Science, Free University of Berlin, Arnimallee 2, D-14195 Berlin, Germany

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ABSTRACT

We prove that the isoperimetric inequalities in the Euclidean and hyperbolic plane hold for all Euclidean, respectively hyperbolic, cone-metrics on a disk with singularities of negative curvature. This is a discrete analog of the theorems of Weil and Bol that deal with Riemannian metrics of curvature bounded from above by 0, respectively by −1. A stronger discrete version was proved by A.D. Alexandrov, with a subsequent extension by approximation to metrics of bounded integral curvature. Our proof uses "discrete conformal deformations" of the metric that eliminate the singularities and increase the area. Therefore it resembles Weil's argument that uses the uniformization theorem and the harmonic minorant of a subharmonic function.

1. Introduction

1.1. The main theorem

A *Euclidean cone-metric* g on a closed surface M is a path metric structure such that every point has a neighborhood isometric either to an open Euclidean disk or to a neighborhood of the apex of a Euclidean cone with angle $\omega \in (0, +\infty) \setminus \{2\pi\}$ around the apex. If M has non-empty boundary, then we require that every boundary point has a neighborhood isometric either to a half-disk or to a circular sector of angle $\theta \in (0, +\infty) \setminus \{\pi\}$. *Hyperbolic cone-metrics* on surfaces are defined similarly. A typical example is the metric space obtained by gluing together Euclidean (respectively hyperbolic) triangles. Conversely, every

* Current address: Department of Mathematics, University of Fribourg, Chemin du Musée 23, CH-1700 Fribourg Pérolles, Switzerland.
E-mail address: ivan.izmestiev@unifr.ch.
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cone-surface can be triangulated so that the metric induced on the triangles is Euclidean, respectively hyperbolic.

The set of cone-like interior and angle-like boundary points is called the *singular locus* of the metric g. An interior cone point with angle ω is said to have *curvature* $2\pi - \omega$.

Theorem 1. *For every Euclidean cone-metric g on a disk $\mathbb{B}^2$ such that all cone points have negative curvatures the following inequality holds:*

$$L^2 \geq 4\pi A \tag{1}$$

where A is the area and L the perimeter of $(\mathbb{B}^2, g)$.

For every hyperbolic cone-metric g on a disk $\mathbb{B}^2$ such that all cone points have negative curvatures the following inequality holds:

$$L^2 \geq 4\pi A + A^2 \tag{2}$$

Inequalities (1), respectively (2) hold for all Euclidean, respectively hyperbolic metrics on a disk, as a consequence of the isoperimetric inequalities in the Euclidean, respectively hyperbolic, plane. Therefore Theorem 1 is implied by the following proposition.

Proposition 2. *For every Euclidean or hyperbolic cone-metric g on $\mathbb{B}^2$ such that all cone points have negative curvatures there is a Euclidean, respectively hyperbolic, metric on $\mathbb{B}^2$ with the same perimeter and larger area.*

Stronger versions of Proposition 2 and Theorem 1 were proved by A.D. Alexandrov, see Section 1.3 below. The aim of the present article is to give a new proof that is simple and in some sense conceptually attractive.

1.2. The generalized Cartan–Hadamard conjecture

Theorem 1 can be viewed as the discrete analog of the following theorem.

Theorem 3. *For every Riemannian metric on a disk $\mathbb{B}^2$ with the Gauss curvature $K(x) \leq 0$ the Euclidean isoperimetric inequality holds.*

For every Riemannian metric on a disk $\mathbb{B}^2$ with $K(x) \leq -1$ the hyperbolic isoperimetric inequality holds.

The first part was proved independently by Weil [15] and by Beckenbach and Radó [2]. The second part is due to Bol [4].

Aubin [1] and Gromov [9] conjectured that a similar result holds in higher dimensions: a simply connected n-manifold with sectional curvature bounded above by κ satisfies the isoperimetric inequality of the space-form with curvature κ. As for now, only the cases $n = 3$ for any $\kappa \leq 0$ [10] and $n = 4$ for $\kappa = 0$ [6] have been verified. See [11] for a novel approach and new results.

1.3. Surfaces of bounded curvature in the sense of Alexandrov

A.D. Alexandrov's stronger version of Theorem 1 is

$$L^2 \geq 2(2\pi - \kappa^+)A - kA^2, \tag{3}$$