Differential Geometry and its Applications 41 (2015) 48–64

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Four-dimensional naturally reductive pseudo-Riemannian spaces ☆

W. Batat [a], M. Castrillón López [b], E. Rosado María [c,*]

[a] École Nationale Polytechnique d'Oran, Département de Mathématiques et Informatique, B.P. 1523, El M'Naouar, Oran, Algeria
[b] ICMAT (CISC, UAM, UC3M, UCM), Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain
[c] Departamento de Matemática Aplicada, E.T.S. Arquitectura, U.P.M., Juan de Herrera 4, 28040 Madrid, Spain

A R T I C L E I N F O

Article history:
Received 12 September 2014
Received in revised form 27 March 2015
Available online 16 May 2015
Communicated by O. Kowalski

MSC:
53C30
53C50

Keywords:
Homogeneous spaces
Naturally reductive
Pseudo-Riemannian spaces

A B S T R A C T

The classification of 4-dimensional naturally reductive pseudo-Riemannian spaces is given. This classification comprises symmetric spaces, the product of 3-dimensional naturally reductive spaces with the real line and new families of indecomposable manifolds which are studied at the end of the article. The oscillator group is also analyzed from the point of view of this classification.

1. Introduction

Homogeneous manifolds play a preeminent role in Differential Geometry and have deserved thorough studies and classifications from different perspectives. Among these spaces, naturally reductive manifolds are possibly the simplest class besides the class of Lie groups or symmetric spaces. This is probably due to the fact that they generalize these spaces in a simple way. Classifications of low dimensional naturally reductive Riemannian homogeneous manifolds can be found in classical references. Beyond the trivial result in surfaces, all connected and simply connected 3-dimensional naturally homogeneous Riemannian spaces are given in [18]: they comprise symmetric spaces together with the Lie groups $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$

☆ The three authors have been partially supported by MICINN, Spain, under grant MTM2011-22528.

* Corresponding author.
E-mail addresses: batatwafa@yahoo.fr (W. Batat), mcastri@mat.ucm.es (M. Castrillón López), eugenia.rosado@upm.es (E. Rosado María).

http://dx.doi.org/10.1016/j.difgeo.2015.04.004
0926-2245/

and the Heisenberg group, endowed with convenient left invariant metrics. The four dimensional case is tackled in [12] where it is proved that under the same topological conditions, a naturally reductive Riemannian 4-manifold necessarily splits as a product of a 3-dimensional naturally reductive manifold and $\mathbb{R}$. We have to wait for the 5-dimensional case to get new indecomposable naturally reductive manifolds (see [13]).

The study of naturally reductive pseudo-Riemannian spaces also deserves special attention. The classification in the 3-dimensional setting has been recently obtained in [4,10] where, again, the manifold is either symmetric, $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ or the Heisenberg group with convenient metrics. The four dimensional case has attired much interest in the literature (see for example [2,16] where the structure of naturally reductive groups are analyzed) probably because of the possible connections of these spaces with plausible relativistic models. The goal of this paper is to provide the complete classification of 4-dimensional naturally reductive pseudo-Riemannian manifolds of $(1, 3)$ or $(2, 2)$ signatures. Surprisingly, the main results (see Theorems 9 and 10) show that, besides the product of a 3-dimensional naturally reductive manifold and $\mathbb{R}$, there is a family of indecomposable manifolds. This situation has no counterpart in the Riemannian case. A previous work [6] has tackled the same problem from a similar point of view to ours. But its classification is not complete as the normal forms for adjoint endomorphism therein considered are those in Proposition 7 b) and Proposition 8 a2) only, missing the rest of the forms of these Propositions, which still give valid cases for the final classification.

The structure of the article is as follows. We first review the basic concepts and properties of naturally reductive manifolds, specially those connected with the notion of homogeneous structure tensors. We then follow the technique of Kowalski and Vanhecke, although we cannot simply generalize [12] due to the existence of the new families mentioned above. At the end of the article, we explore the geometry of these new manifolds to be sure that they are indecomposable and non-symmetric. Finally, we apply Theorem 9 to the analysis of the 4-dimensional oscillator group, probably the most relevant naturally reductive Lorentzian example in the literature. We give a decomposition of this space which is different to the one of its traditional definition.

2. Preliminaries

2.1. Naturally reductive spaces

Let (M, g) be a reductive pseudo-Riemannian homogeneous manifold of dimension n. This means that $M = G/H$, where G is connected Lie group of isometries acting transitively and effectively on M, H is the isotropy of a point $o \in M$, and the Lie algebra $\mathfrak{g}$ of G admits a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. The mapping $A \mapsto A^*_o = d/d\varepsilon|_{\varepsilon=0} \exp(\varepsilon A) \cdot o$ defines an isomorphism between $\mathfrak{m}$ and T_oM which, in addition, is used to transfer the metric g to $\mathfrak{m}$. For convenience, along the article we will denote both the metric in T_oM and in $\mathfrak{m}$ by $\langle\cdot,\cdot\rangle$. The decomposition of $\mathfrak{g}$ is said to be naturally reductive if in addition

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle [X, Z]_{\mathfrak{m}}, Y\rangle = 0 \text{ for } X, Y, Z \in \mathfrak{m}, \tag{1}$$

where $[\cdot,\cdot]_{\mathfrak{m}}$ is the $\mathfrak{m}$-part of the bracket (see, e.g., [11, Chapter X, section 3], [15, Chapter 11, Definition 23]). Let $\widetilde{\nabla}$ be the canonical connection of the reductive homogeneous space $M = G/H$. It is well known that the torsion tensor $\tilde{T}$ and the curvature tensor $\tilde{R}$ of $\widetilde{\nabla}$ at the point o read

$$\widetilde{T}(X, Y)_o = -[X, Y]_{\mathfrak{m}}, \quad \widetilde{R}(X, Y)_o = -[X, Y]_{\mathfrak{h}}, \quad \forall X, Y \in \mathfrak{m}. \tag{2}$$

Recalling that G-invariant tensor fields on M are parallel with respect to the connection $\widetilde{\nabla}$, we have