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Expansive flows of the three-sphere

Alfonso Artigue [1]

Departamento de Matemática y Estadística del Litoral, Universidad de la República, Gral. Rivera 1350, Salto, Uruguay

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ABSTRACT

In this article we show that the three-dimensional sphere admits transitive expansive flows in the sense of Komuro with hyperbolic equilibrium points. The result is based on a construction that allows us to see the geodesic flow of a hyperbolic three-punctured two-dimensional sphere as the flow of a smooth vector field on the three-dimensional sphere.

1. Introduction

In the study of dynamical systems several authors considered the problem of determining which compact manifolds M admit expansive systems. Let us recall that in the discrete-time setting $f\colon M \to M$ is an *expansive homeomorphism* [20] if there is $\delta > 0$ such that $\operatorname{dist}(f^n(x), f^n(y)) < \delta$ for all $n \in \mathbb{Z}$ implies $x = y$. In [12] it is proved that the circle does not admit expansive homeomorphisms. In [18] it is shown that every orientable compact surface of positive genus admits expansive homeomorphisms, namely, a pseudo-Anosov diffeomorphism. In [11,15] it is proved that the two-sphere does not admit expansive homeomorphisms. They also proved that every expansive surface homeomorphism is conjugate to a pseudo-Anosov diffeomorphism. This completes a global picture of expansive homeomorphisms of orientable compact surfaces.

In higher dimensions there is no result characterizing which manifolds admit expansive homeomorphisms. Let us mention some advances in this direction. In [10] it is proved that expansive homeomorphisms of tori

E-mail address: artigue@unorte.edu.uy.

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with the pseudo-orbit tracing property are conjugate to hyperbolic automorphisms. In [21,22] it is proved that an expansive homeomorphism of a compact three-dimensional manifold with a dense set of topologically hyperbolic periodic points is conjugate to a linear Anosov isomorphism on the torus. In [2] this result was generalized for arbitrary dimension assuming the existence of a codimension one periodic point. In [23] it is proved that expansive $C^{1+\theta}$-diffeomorphisms on three-manifolds without wandering points are conjugate to Anosov diffeomorphisms on the torus. The main difficulty, from our viewpoint, for classifying expansive homeomorphisms of three-manifolds is to understand the topology of local stable and unstable sets. To our best knowledge it is unknown whether the sphere $\mathbb{S}^3$ admits an expansive homeomorphism.

For the case of vector fields or flows the corresponding problems are considered. According to Bowen and Walters [6] we say that $\phi\colon \mathbb{R} \times M \to M$ is an *expansive flow* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ being $h\colon \mathbb{R} \to \mathbb{R}$ a parameterization, i.e., an increasing homeomorphism with $h(0) = 0$, then $y = \phi_s(x)$ for some $s \in (-\varepsilon, \varepsilon)$. An important fact about this definition is that it does not allow singular (or equilibrium) points. Trivially, every circle flow without singular points is expansive. It is known that no compact surface admits an expansive flow [16,3,7] in the sense of [6]. In [19] it is shown that if a compact three-manifold admits an expansive flow then its fundamental group has exponential growth. In particular, the three-sphere does not admit expansive flows in the sense of Bowen and Walters.

The expansiveness of flows with singular points was first investigated in [14]. In this paper Komuro proved that the Lorenz attractor is k^*-expansive, a definition designed to allow singularities. According to [14], a flow is k^*-*expansive* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ being h a reparameterization, then $\phi_{h(t_0)}(x) = \phi_{t_0+s}(y)$ for some $s \in (-\varepsilon, \varepsilon)$ and $t_0 \in \mathbb{R}$. In [3] it is proved that a flow is k^*-expansive if and only if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\operatorname{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ and a reparameterization h, then there is $z \in M$ such that $x, y \in \phi_{[0,\varepsilon]}(z)$ and $\operatorname{diam}(\phi_{[0,\varepsilon]}(z)) < \varepsilon$. It is easy to see that a circle flow is k^*-expansive if and only if it has a finite number of singularities. In [3] it is shown that every k^*-expansive surface flow is obtained from surgery on the suspension of minimal interval exchange maps. It is also proved that a surface admits a k^*-expansive flow if and only if it is a two-torus with b boundary components, h handles and c cross-cups with $b + h + c > 0$. In particular the two-torus and the two-sphere do not admit k^*-expansive flows.

From the definitions it is easy to see that every expansive flow in the sense of Bowen and Walters is k^*-expansive. In order to obtain k^*-expansive flows with singular points on a manifold with dimension greater than 2 we can proceed as follows. Take M admitting a Bowen–Walters expansive flow generated by a vector field X. Let $\rho\colon M \to \mathbb{R}$ be a non-negative smooth function vanishing only at $p \in M$. The vector field ρX generates a k^*-expansive flow with a zero-index singular point p. These kind of points are usually called *fake singularities*. No published example of a k^*-expansive flow with hyperbolic singularities on a manifold of dimension greater than 2 is known to the author. This kind of expansiveness was deeply studied in relation with singular hyperbolic vector fields and three-dimensional attractors, as for example the one discovered by Lorenz, see [1]. Also, the concept of sectional-Anosov flow seems to be related with k^*-expansivity [4].

The purpose of the present paper is to show that the three-sphere $\mathbb{S}^3$ admits k^*-expansive flows. Let us sketch the construction while describing the contents of the article. In Section 2 we will consider a triangular billiard in the hyperbolic disc, i.e., the curvature of the surface is -1 and the boundary consists of three geodesic arcs. This dynamical system is related with the geodesic flow of a two-sphere with three punctures. The unit tangent bundle of this three-punctured sphere will be embedded in a closed three-manifold M. In Section 3 it is shown that M is homeomorphic to $\mathbb{S}^3$. In Section 4 we will show that a reparameterization of the geodesic flow of the three-punctured sphere can be extended to the whole M. In Section 5 we show that this extended flow is k^*-expansive. In Corollary 5.2 we deduce that the three-sphere admits a transitive k^*-expansive flow with a dense set of periodic orbits.

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