Differential Geometry and its Applications 40 (2015) 1–13

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Some aspects of Dirac-harmonic maps with curvature term

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ARTICLE INFO

Article history:
Received 13 January 2015
Available online 2 March 2015
Communicated by Th. Friedrich

MSC:
53C27
58E20
35J61

Keywords:
Dirac-harmonic map with curvature term
Regularity
Vanishing theorem

ABSTRACT

We study several geometric and analytic aspects of Dirac-harmonic maps with curvature term from closed Riemannian surfaces.

1. Introduction and results

Dirac-harmonic maps arise as critical points of part of the nonlinear σ-model studied in quantum field theory [12]. They form a pair of a map from a Riemann surface to a Riemannian manifold and a vector spinor. The equations for Dirac-harmonic maps couple the harmonic map equation to spinor fields. As limiting cases both harmonic maps and harmonic spinors can be obtained. Moreover, Dirac-harmonic maps belong to the class of conformally invariant variational problems and thus have a lot of nice properties.

Many important results for Dirac-harmonic maps have already been established. This includes several analytical results [11,23,14,26] and an existence result for uncoupled solutions [2]. The boundary value problem for Dirac-harmonic maps is discussed in [10]. A heat-flow approach to Dirac-harmonic maps was studied recently in [4,7].

However, the full nonlinear supersymmetric σ-model in physics contains additional terms, which are not captured by the analysis of Dirac-harmonic maps, see [15,8] for the physical background. Taking into account

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http://dx.doi.org/10.1016/j.difgeo.2015.01.008
0926-2245/

and additional two-form in the action functional the resulting equations were studied in [6], Dirac-harmonic maps to target spaces with torsion are analyzed in [5].

In this article we focus on Dirac-harmonic maps coupled to a curvature term, which were introduced in [9]. This set of equations also has an interesting limit. In the case of the map part being trivial it reduces to a nonlinear Dirac equation, which was studied in [13] and [22]. Moreover, it should be noted that this equation also appears in the context of the spinorial representation of surfaces in $\mathbb{R}^3$ [16] and the Thirring model in quantum field theory [21].

In the general case Dirac-harmonic maps with curvature term are more complicated then Dirac-harmonic maps since they consist of a pair of two non-linear equations.

The aim of this article is to establish some basic results for Dirac-harmonic maps with curvature term, in particular the regularity of weak solutions.

This article is organized as follows. In Section 2 we recall the notion of Dirac-harmonic maps with curvature term. Section 3 discusses geometric and Section 4 analytical properties of Dirac-harmonic maps with curvature term.

2. Dirac-harmonic maps with curvature term

Let us now describe the setup in more detail. For a map $\phi\colon M \to N$ we study its differential $d\phi \in \Gamma(T^*M \otimes \phi^{-1}TN)$, integrating the square of its norm leads to the usual harmonic energy. We assume that (M,h) is a closed Riemannian spin surface with spinor bundle ΣM, for more details about spin geometry see the book [18]. Moreover, let (N,g) be another closed Riemannian manifold. Together with the pullback bundle $\phi^{-1}TN$ we consider the twisted bundle $\Sigma M \otimes \phi^{-1}TN$. The induced connection on this bundle will be denoted by $\tilde{\nabla}$. Sections $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ in this bundle are called *vector spinors* and the natural operator acting on them is the twisted Dirac operator, denoted by $\not{D}$. It is an elliptic, first order operator, which is self-adjoint with respect to the L^2-norm. More precisely, the twisted Dirac operator is given by $\not{D} = e_\alpha \cdot \tilde{\nabla}_{e_\alpha}$, where $\{e_\alpha\}$ is an orthonormal basis of TM and $\cdot$ denotes Clifford multiplication. We are using the Einstein summation convention, that is we sum over repeated indices. Clifford multiplication is skew-symmetric, namely

$$\langle \chi, X \cdot \xi \rangle_{\Sigma M} = -\langle X \cdot \chi, \xi \rangle_{\Sigma M}$$

for all $\chi, \xi \in \Gamma(\Sigma M)$ and all $X \in TM$. In addition, the Clifford relations

$$X \cdot Y + Y \cdot X = -2h(X,Y)$$

hold for all $X, Y \in TM$.

We use Greek letters for indices on M and Latin letters for indices on N. In terms of local coordinates y^i the vector spinor ψ can be written as $\psi = \psi^i \otimes \frac{\partial}{\partial y^i}$, and thus the twisted Dirac operator $\not{D}$ is locally given by

$$\not{D}\psi = \left(\not{\partial}\psi^i + \Gamma^i_{jk} \nabla\phi^j \cdot \psi^k\right) \otimes \frac{\partial}{\partial y^i}.$$

Here, $\not{\partial}\colon \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ denotes the usual Dirac operator. Throughout this article we study the functional

$$E_c(\phi,\psi) = \frac{1}{2}\int_M |d\phi|^2 + \langle \psi, \not{D}\psi \rangle - \frac{1}{6}\langle R^N(\psi,\psi)\psi, \psi \rangle. \tag{2.1}$$