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Higher holonomies: Comparing two constructions

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ABSTRACT

We compare two different constructions of higher-dimensional parallel transport. On the one hand, there is the two-dimensional parallel transport associated with 2-connections on 2-bundles studied by Baez–Schreiber [2], Faria Martins–Picken [11] and Schreiber–Waldorf [12]. On the other hand, there are the higher holonomies associated with flat superconnections as studied by Igusa [7], Block–Smith [3] and Arias Abad–Schätz [1]. We first explain how by truncating the latter construction one obtains examples of the former. Then we prove that the two-dimensional holonomies provided by the two approaches coincide.

1. Introduction

The purpose of this note is to compare two extensions of parallel transport to higher-dimensional objects. On the one hand, there is the parallel transport for flat 2-connections with values in crossed modules studied by Baez–Schreiber [2], Faria Martins–Picken [11] and Schreiber–Waldorf [12]. The fundamental result in this approach is the construction of two-dimensional holonomies. This construction yields a map

$$\mathsf{Hol} : \mathsf{Flat}(M, \mathsf{g}) \to \mathrm{Rep}(\pi_{\leq 2}(M), \mathsf{G}),$$

that assigns to any flat 2-connection with values in the differential crossed module g a representation of the fundamental 2-groupoid of M. Here G is a Lie crossed module whose infinitesimal counterpart is g.

On the other hand, there is the parallel transport of flat superconnections introduced recently by Igusa [7] and subsequently studied by Block–Smith [3] and Arias Abad–Schätz [1]. The fundamental result in this direction is that there is a weak equivalence of *dg*-categories between the category $\mathsf{Rep}_\infty(TM)$ of flat

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superconnections on M and the category of ∞-representations of the ∞-groupoid $\mathsf{Rep}_\infty(\pi_\infty(M))$ of M. In particular, there is an integration A_∞-functor

$$\int : \mathsf{Rep}_\infty(TM) \to \mathsf{Rep}_\infty(\pi_\infty(M)),$$

which associates to any flat superconnection on M an ∞-representation of the ∞-groupoid of M. This integration procedure can be understood as a consequence of Gugenheim's A_∞-version of de Rham's theorem [6].

We consider flat connections with values in a fixed finite-dimensional complex (V, ∂), i.e. V is concentrated in finitely many degrees and each of its homogeneous components is finite-dimensional. The integration functor $\int$ restricts to a functor between the resulting full subcategories of $\mathsf{Rep}_\infty(TM)$ and $\mathsf{Rep}_\infty(\pi_\infty(M))$, denoted by $\mathsf{Rep}_\infty(TM, V)$ and $\mathsf{Rep}_\infty(\pi_\infty(M), V)$, respectively. Given a flat superconnection α with values in (V, ∂), the integration functor associates holonomies to any simplex in M. By construction, the holonomy associated with a path γ – seen as a 1-simplex – coincides with ordinary parallel transport along γ and yields an automorphism of (V, ∂). The holonomy $\mathsf{Hol}(\sigma)$ of an n-dimensional simplex σ is a linear endomorphism of V of degree $1 - n$. The fact that holonomies are built coherently is formalized by expressing the commutator $[\partial, \mathsf{Hol}(\sigma)]$ in terms of the holonomies associated with subsimplices of σ.

For the purpose of this paper we want to focus on one- and two-dimensional holonomies. To this end, one factors out all the information related to simplices of dimension strictly larger than 2. More formally, the flat superconnection α is a sum of components of fixed form-degree and we disregard all components of form-degree strictly larger than 2. Similarly, one can truncate the ∞-groupoid $\pi_\infty(M)$ of M which leads to the fundamental 2-groupoid $\pi_{\leq 2}(M)$ mentioned above. Observe that, formally, we also ignore the categorical structure on $\mathsf{Rep}_\infty(TM, V)$ and $\mathsf{Rep}_\infty(\pi_\infty(M), V)$ respectively, since we work on the level of objects only. We indicate this transition with the change of notation from Rep_∞ to Rep_∞.

As mentioned above, the holonomies associated with paths are elements of the automorphism group of (V, ∂). Hence one expects that the holonomies associated with 2-simplices should belong to the automorphism 2-group of (V, ∂). One way to make this precise is to consider the Lie crossed module $\mathsf{GL}(V)$ associated with V and its infinitesimal version $\mathsf{gl}(V)$. We show that the truncation of a flat superconnection can be seen as a flat 2-connection on M with values in $\mathsf{gl}(V)$ and that the 2-truncation of $\int \alpha \in \mathrm{Rep}_\infty(\pi_\infty(M), V)$ is a 2-representation of $\pi_{\leq 2}(M)$ on (V, ∂). Hence, we obtain a diagram of the form

$$\begin{array}{ccc}
\mathrm{Rep}_\infty(TM, V) & \xrightarrow{\ \int\ } & \mathrm{Rep}_\infty(\pi_\infty(M), V) \\
\downarrow{\scriptstyle \mathsf{T}_{\leq 2}} & & \downarrow{\scriptstyle \mathsf{T}_{\leq 2}} \\
\mathsf{Flat}(M, \mathsf{gl}(V)) & & \mathrm{Rep}(\pi_{\leq 2}(M), \mathsf{GL}(V)).
\end{array}$$

This allows us to compare the integration functor $\int$ to the two-dimensional holonomies for flat 2-connections with values in a differential Lie crossed module, which specializes to

$$\mathsf{Hol} : \mathsf{Flat}(M, \mathsf{gl}(V)) \to \mathrm{Rep}(\pi_{\leq 2}(M), \mathsf{GL}(V)).$$

Our main result is that the above two diagrams are compatible, i.e. the 2-truncation of $\int \alpha$ actually coincides with the holonomy construction proposed in [2,11,12]. More precisely:

Theorem 4.8. *Let M be a smooth manifold and (V, ∂) a cochain complex of vector spaces of finite type. Then the following diagram commutes:*