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Uniform bounds for the heat content of open sets in Euclidean space ☆

M. van den Berg *, K. Gittins

School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK

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A B S T R A C T

We obtain (i) lower and upper bounds for the heat content of an open set in $\mathbb{R}^m$ with R-smooth boundary and finite Lebesgue measure, (ii) a necessary and sufficient geometric condition for finiteness of the heat content in $\mathbb{R}^m$, and corresponding lower and upper bounds, (iii) lower and upper bounds for the heat loss of an open set in $\mathbb{R}^m$ with finite Lebesgue measure.

1. Introduction

In this paper we obtain results for the heat content of an open and bounded set D in Euclidean space $\mathbb{R}^m$, $m = 2, 3, \cdots$, where D has initial temperature 1 and the complement of D has initial temperature 0. We denote the fundamental solution of the heat equation on $\mathbb{R}^m$ by

$$p(x, y; t) = (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)},$$

and define

$$u_D(x; t) = \int_D dy\, p(x, y; t). \tag{1}$$

It is standard to check (see Chapter 2 in [12]) that

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* Corresponding author.
E-mail address: mamvdb@bristol.ac.uk (M. van den Berg).

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$$\Delta u_D = \frac{\partial u_D}{\partial t}, \quad x \in \mathbb{R}^m, \, t > 0, \tag{2}$$

and that

$$\lim_{t \downarrow 0} u_D(x;t) = \mathbb{1}_D(x), \quad x \in \mathbb{R}^m - \partial D, \tag{3}$$

where ∂D is the boundary of D.

We define the *heat content of D in $\mathbb{R}^m$ at t* by

$$H_D(t) = \int_D dx \, u_D(x;t). \tag{4}$$

Thus, if $u_D(x;t)$ represents the temperature at point $x \in \mathbb{R}^m$ at time t with initial condition (3), then the heat content of D in $\mathbb{R}^m$ at time t represents the amount of heat in D at time t. By (1) and (4) we see that

$$H_D(t) = \int_D dx \int_D dy \, p(x,y;t). \tag{5}$$

By the heat semigroup property we have that

$$p(x,y;t) = \int_{\mathbb{R}^m} dz \, p(x,z;t/2)p(z,y;t/2). \tag{6}$$

By Tonelli's Theorem and (1), (5) and (6) we conclude that

$$H_D(t) = \|u_D(\cdot;t/2)\|^2_{L^2(\mathbb{R}^m)}.$$

Preunkert [15] defines the L^2-curve of the set D as the map $t \mapsto \|u_D(\cdot;t/2)\|_{L^2(\mathbb{R}^m)}$. The results for the L^2-curve in Theorem 2.4 of [16] imply that if D is an open, bounded subset of $\mathbb{R}^m$ with $C^{1,1}$-boundary ∂D then

$$H_D(t) = |D| - \pi^{-1/2}\mathcal{P}(D)t^{1/2} + o(t^{1/2}), \; t \downarrow 0, \tag{7}$$

where $|D|$ denotes the Lebesgue measure of D and $\mathcal{P}(D)$ denotes the perimeter of D. Note that since ∂D is Lipschitz, $\mathcal{P}(D) = \mathcal{H}^{m-1}(\partial D)$, the $(m-1)$-dimensional Hausdorff measure of the boundary (see Remark (ii) on p. 183 in [13]).

Initial value problems of the type (2)–(3) have been studied in the much wider context of operators of Laplace type on compact Riemannian manifolds [8]. The results of that paper imply that if D is open, bounded in $\mathbb{R}^m$ with C^∞ boundary then there exist geometric invariant $h_0, h_1, \cdots$ such that for any $J \in \mathbb{N}$,

$$H_D(t) = \sum_{j=0}^{J} h_j t^{j/2} + O(t^{(J+1)/2}), \;\; t \downarrow 0. \tag{8}$$

Furthermore if ∂D is oriented by a unit inward pointing normal vector field and if $\{k_1(s), \cdots, k_{m-1}(s)\}$ are the principal curvatures at $s \in \partial D$ then $h_0 = |D|$, $h_1 = -\pi^{-1/2} \int_{\partial D} d\mathcal{H}^{m-1}(s)$, $h_2 = 0$ and

$$h_3 = -\int_{\partial D} d\mathcal{H}^{m-1}(s) \left(\frac{5}{32} \left(\sum_{i=1}^{m-1} k_i(s) \right)^2 + \frac{1}{16} \sum_{i=1}^{m-1} k_i^2(s) \right).$$

For further results in the Riemannian manifold setting we refer to [11] and [9].