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Conformally flat circle bundles over surfaces

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ABSTRACT

We consider Riemannian 3-manifolds P which admit a free isometric circle action and we compute the equations of conformal flatness of P in terms of the geometry of the fibration $P \to P/S^1$. This computation is applied in order to classify conformally flat circle bundles over compact oriented surfaces.

1. Introduction

We consider compact oriented conformally flat Riemannian 3-manifolds P such that there exists a free isometric circle action $P \times S^1 \to P$. This gives rise to a principal S^1-bundle $P \to M$ over a Riemannian surface M with principal connection. The condition that P is conformally flat can be written as a differential equation on M in terms of the curvature function H of this connection and the Gaussian curvature K of M:

$$Hess\, H = H(H^2 - K)Id, \tag{1}$$

$$2K - 3H^2 = \alpha \tag{2}$$

for some constant α. These equations are strongly related to the geometry of the surface. For example, integral curves of $grad\, H$ are geodesics. We prove that around a regular critical point p of H, H itself depends only on the geodesic distance $r = d(., p)$ from p. The function H satisfies the equation $cH'(r) = L(r)$, where c is some constant and $L(r)$ is the length of the circle of distance r around p. Applying this observation we deduce that the curvature functions H and K must be constant. This enables us to give a full classification of conformally flat circle bundles over compact surfaces.

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2. Circle bundles

Definition 1. A Riemannian manifold (P, g), together with a submersion $\pi: P \to M$, is called circle bundle with circle metric g if it is a principal bundle $S^1 \to P \to M$ such that S^1 acts by isometries.

The proposition below characterizes Riemannian manifolds P occurring as total spaces of circle bundles.

Proposition 2. *Every compact, oriented Riemannian 3-manifold P for which a Riemannian submersion $\pi: P \to M$ to an oriented surface M with connected minimal fibers does exist, possesses a free isometric circle action $P \times S^1 \to P$, and vice versa.*

Let us first fix some notations. Let T be always the vector field in positive fiber direction of constant length 1 on the total space P. We will consider (locally defined) positive oriented orthonormal vector fields A, B on the base M, and denote their horizontal lifts on P by $\hat{A}, \hat{B}$. It is a basic fact that there exists function λ such that $e^\lambda(A - iB)$ is a (locally defined) holomorphic vector field on the surface. Note that $e^\lambda(A - iB)$ is holomorphic if and only if $[A, B] = B \cdot \lambda A - A \cdot \lambda B$.

The horizontal distribution given by $\mathcal{H} = \ker d\pi^\perp$ is invariant under the S^1-action and gives rise to a principal connection ω. The curvature Ω of this connection is an imaginary valued 2-form which is invariant under the S^1 action. Therefore, the function H defined by

$$\Omega = iH\pi^* vol_M$$

is constant along the fibers. We denote the corresponding function on the surface by H, too. Let K the Gaussian curvature (function) of the surface M with respect to the induced Riemannian metric h on M. For stating the formulas below we will use the endomorphism $\mathcal{J} \in End(TP)$ given by $X \mapsto T \times X$.

Proposition 3. *Let $\hat{A}, \hat{B}, T, \lambda, H$ be given as above. The Levi-Civita connection of the total space of a circle bundle $P \to M$ with circle metric g is determined by*

$$\nabla T = \frac{1}{2} H \mathcal{J} \tag{3}$$

$$\nabla A = B \otimes g(., \frac{1}{2} HT + \mathcal{J} grad(\lambda \circ \pi)) + T \otimes g(., \frac{1}{2} HB) \tag{4}$$

$$\nabla B = -A \otimes g(., \frac{1}{2} HT + \mathcal{J} grad(\lambda \circ \pi)) - T \otimes g(., \frac{1}{2} HA). \tag{5}$$

It is well-known that in dimension 3 the Riemannian curvature tensor R is entirely given by the Ricci tensor *Ric*. In our situation it is more convenient to work with the so-called Schouten tensor

$$S := Ric - \frac{1}{4} scal Id \in End(TP).$$

Proposition 4. *The Schouten tensor of the total space of a circle bundle $P \to M$ with circle metric g is given by*

$$S(T, T) = (-\frac{1}{2} K + \frac{5}{8} H^2)$$

$$S(T, X) = g(-\frac{1}{2} \mathcal{J} grad\, H, X)$$

$$S(X, Y) = (-\frac{3}{8} H^2 + \frac{1}{2} K) g(X, Y),$$

where X and Y are arbitrary horizontal vectors and $T, H,$ and K are defined as above.