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On the fundamental equations of homogeneous Finsler spaces ☆

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ABSTRACT

By introducing the notion of single colored Finsler manifold, we deduce the curvature formulas of a homogeneous Finsler space. It results in a set of fundamental equations that are more elegant than the Riemannian case. Several applications of the equations are also supplied.

1. Introduction

The primary goal of this paper is to provide a direct method for discussing curvatures of homogeneous Finsler spaces. A remarkable fact in homogeneous Finsler geometry is that, the computation of curvature itself is one of the major difficulties in the study of various curvature problems. So far only a few cases are known to be computable: naturally reductive Finsler metrics [6] are known to share the curvature tensor with some Riemannian metrics; Randers metrics are computable [8] though the method and resulting formulas are tedious. This phenomenon is completely different from the Riemannian case. In Riemannian geometry, the curvature formula for a homogeneous space is probably the simplest among all important examples.

To the author's knowledge, there are three methods to deduce the curvature formula in Riemannian case. The first one appears in K. Nomizu [16] as a byproduct of his invariant connection theory (note that there is a sign error in Nomizu's formula). The second one is invented and used by S. Helgason [7], it is unusual and complicated. The third treatment is very popular today (see [3]), it heavily depends on the property of Killing fields. All these methods are not easy to adopt to Finsler case, at least so far have not been

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successful. The method offered below is based on the consideration of single colored Finsler manifolds (for definition, see Section 3.1), which may also prove useful in other topics in Finsler geometry.

Prof. Zhongmin Shen has said, in several occasions, that a Finsler manifold is usually *colorful*, because if we assign a color to each kind of Minkowski space, then a Finsler manifold can admit many colors. It is thus an interesting subject to study those Finsler manifolds with a single color.

It turns out that the curvatures of a single colored Finsler manifold consists of two sets of data. One comes from a single Minkowski norm and the other comes from the manifold itself. Applying this idea to the homogeneous case, we find that there are two quantities which play crucial role in all the curvatures. One is the spray vector η, the other is the connection operator N. Using these two quantities, one can compute various kinds of curvatures of a homogeneous space, such as flag curvature, Ricci curvature, Landsberg curvature, S curvature, etc. For example, the Riemann curvature tensor satisfies the following elegant equation

$$g_y(R_y(v), v) = g_y([[v, y]_{\mathfrak{h}}, v], y) + g_y(\tilde{R}(v), v), \quad v \in \mathfrak{m},$$

where $\tilde{R} = D_\eta N - N^2 + [N, \mathrm{ad}_{\mathfrak{m}}(y)]$. For the notations and related discussion, one is referred to Section 4. The above equation, together with other curvature formulas, will be called fundamental equations of a homogeneous space.

The second goal of this paper is to present some basic applications of these equations. For example, we shall give a simple lemma to describe some flags with non-negative flag curvature. It generalizes Lemma 4.1 in [10] and also Theorem 4.7 in [8]. Using this lemma, we give an easier proof to a theorem in [8,14].

Now we briefly describe the organization of this paper. Section 2 is a quick introduction to some basic facts on Finsler geometry. Section 3 is devoted to the investigation of single colored Finsler manifolds. Section 4 gives a detailed deduction of the fundamental equations of a homogeneous Finsler space. Finally, in Section 5, we give several applications of these equations.

We hope that the investigation here will give new insights to the study of Finsler geometry and homogeneous spaces.

2. Preliminaries

This section is mainly to recall some basic facts on Finsler geometry and to fix notations. In this quick tour to Finsler geometry, we will follow the conventions of Chern–Shen's textbook [4] and sometimes [2].

2.1. Minkowski norms and Finsler metrics

Definition 1. Let V be a real vector space of dimension m. A smooth function $F : V\backslash\{0\} \to \mathbb{R}^+$ is called a *Minkowski norm*, if the following conditions are satisfied

1. $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in V\backslash\{0\}$.
2. For each fixed $y \in V\backslash\{0\}$, the bilinear function $g_y : V \times V \to \mathbb{R}$ defined by

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(y + su + tv)\bigg|_{s=t=0}$$

is an inner product on V.

The linear space V equiped with a Minkowski norm F is called a *Minkowski space*, denoted by (V, F).