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Hamiltonian minimality of normal bundles over the isoparametric submanifolds

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ABSTRACT

Let N be a complex flag manifold of a compact semi-simple Lie group G, which is standardly embedded in the Lie algebra $\mathfrak{g}$ of G as a principal orbit of the adjoint action. We show that the normal bundle of N in $\mathfrak{g}$ is a Hamiltonian minimal Lagrangian submanifold in the tangent space $T\mathfrak{g}$ which is naturally regarded as the complex Euclidean space. Moreover, we specify the complex flag manifolds with this property in the class of full irreducible isoparametric submanifolds in the Euclidean space.

1. Introduction and main results

A Lagrangian submanifold L is an m-dimensional submanifold in a $2m$-dimensional symplectic manifold (M,ω) on which the pull-back of the symplectic form ω vanishes. When M is a Kähler manifold, extrinsic properties of Lagrangian submanifolds have been studied by many authors. For instance, Harvey and Lawson [10] established the calibrated geometry, and introduced the notion of special Lagrangian submanifolds in Ricci-flat Kähler manifolds. These are calibrated submanifolds, and automatically volume-minimizing in each homology class. Therefore, they are stable in the classical sense in the volume variational problem.

Since the Lagrangian property is preserved by Hamiltonian flows, it is natural to consider the variational problem under compactly supported Hamiltonian deformations. A Lagrangian submanifold which attains an extremal of the volume functional under Hamiltonian deformations is called *Hamiltonian minimal* (shortly, H-minimal). This was first investigated by Y.-G. Oh [18], where he gave some basic examples. Many more examples have been constructed in Kähler manifolds by various methods (see [2] and references therein).

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When the ambient Kähler manifold is the complex Euclidean space $\mathbb{C}^m$, Y.-G. Oh [18] pointed out that the standard tori $T^m = S^1(r_1) \times \cdots \times S^1(r_m)$ are H-minimal. Generalizing Oh's results [18], Y. Dong [9] showed that the pre-image of an H-minimal Lagrangian submanifold in the complex projective space $\mathbb{C}P^{m-1}$ via the Hopf fibration $\pi : S^{2m-1} \to \mathbb{C}P^{m-1}$ is H-minimal Lagrangian in $\mathbb{C}^m$. We note that there are some known H-minimal Lagrangian submanifolds in $\mathbb{C}P^{m-1}$. For instance, any compact, extrinsically homogeneous Lagrangian submanifolds in $\mathbb{C}P^{m-1}$ are H-minimal, and Bedulli and Gori [4] give the complete classification of Lagrangian orbits which are obtained by a simple Lie group of isometries acting on $\mathbb{C}P^{m-1}$. On the other hand, Anciaux and Castro [2] gave examples of H-minimal Lagrangian immersions of manifolds with various topologies by taking a product of a Lagrangian surface and Legendrian immersions in odd-dimensional unit spheres. These are compact and contained in a sphere. In the present paper, we give a new family of non-minimal, H-minimal Lagrangian submanifolds in $\mathbb{C}^m$, which are non-compact, and most of them are complete and have some symmetries.

In the following, we always identify the tangent space $T\mathbb{R}^{n+k}$ of the Euclidean space $\mathbb{R}^{n+k}$ with $\mathbb{C}^{n+k}$. Let N^n be a submanifold in $\mathbb{R}^{n+k}$. Our examples are given by the normal bundle νN of N in $T\mathbb{R}^{n+k}$. It is known that the normal bundle νN is a Lagrangian submanifold in $T\mathbb{R}^{n+k}$. Harvey and Lawson [10] first showed that νN is a minimal Lagrangian submanifold if and only if N is an austere submanifold, namely, the set of principal curvatures of N with respect to any unit normal vector is invariant under the multiplication by -1. In their context, the condition that a Lagrangian submanifold is minimal is equivalent to that it is a special Lagrangian submanifold of some phase. Hence, one can construct examples of special Lagrangian submanifold in $\mathbb{C}^{n+k}$ from austere submanifolds.

For the H-minimality of normal bundles, we show the following:

Theorem 1.1. *Let G be a compact, connected, semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G with an $\mathrm{Ad}(G)$-invariant inner product, and $N^n = \mathrm{Ad}(G)w$ a principal orbit of the adjoint action of G on $\mathfrak{g} \simeq \mathbb{R}^{n+k}$ through $w \in \mathfrak{g}$. Then the normal bundle νN of N is an H-minimal Lagrangian submanifold in the tangent bundle $T\mathfrak{g} \simeq \mathbb{C}^{n+k}$.*

The principal orbit N is diffeomorphic to G/T, where T is a maximal torus of G, and N is called a *complex flag manifold* or *regular Kähler C-space*. Since $N = \mathrm{Ad}(G)w$ is compact, N is never austere in $\mathbb{R}^{n+k}$, and hence, νN is not minimal. Moreover, it does not have parallel mean curvature vector (see Proposition 2.4). We also note that the normal bundle of $N = \mathrm{Ad}(G)w$ is always trivial, namely, νN is homeomorphic to $N \times \mathbb{R}^k$.

The principal orbits of the adjoint action of a compact semisimple Lie group G on $\mathfrak{g}$ are known as examples of the *isoparametric submanifolds*, namely, submanifolds in $\mathbb{R}^{n+k}$ with flat normal bundles and constant principal curvatures (see Section 3). In the class of isoparametric submanifolds, we show that the complex flag manifolds are essentially only examples which have H-minimal normal bundles. More precisely, we prove the following:

Theorem 1.2. *Let N be a full, irreducible isoparametric submanifold in the Euclidean space $\mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ if and only if N is a principal orbit of the adjoint action of a compact simple Lie group G.*

Any normal bundle of such orbit is complete. We also give examples of non-complete H-minimal Lagrangian varieties as the twisted normal cones over the isoparametric hypersurfaces in the sphere (see Section 5).

The strategy of the proof of Theorems 1.1 and 1.2 is as follows. The mean curvature form $\alpha_{\tilde{H}}$ of a Lagrangian submanifold L in $\mathbb{C}^{n+k}$ is given by $\alpha_{\tilde{H}} = -d\theta$, where θ is an S^1-valued function on L, called the Lagrangian angle. It is known that the H-minimality is equivalent to the differential equation $\Delta\theta = 0$ on L. We calculate θ of the normal bundle νN of a submanifold N in $\mathbb{R}^{n+k}$, and show that the angle is expressed