Differential Geometry and its Applications 33 (2014) 4–43

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Conformal holonomy, symmetric spaces, and skew symmetric torsion ☆

Jesse Alt [a], Antonio J. Di Scala [b], Thomas Leistner [c,*]

[a] *School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa*
[b] *Dipartimento di Scienze Matematiche "G.L. Lagrange", Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129, Torino, Italy*
[c] *School of Mathematical Sciences, University of Adelaide, SA 5005, Australia*

ARTICLE INFO

Article history:
Available online 12 November 2013
Communicated by A. Čap

Dedicated to Michael G. Eastwood on the occasion of his 60th birthday

MSC:
primary 53A30, 53C29
secondary 53C35, 53C50

Keywords:
Conformal holonomy
Symmetric spaces
Nearly para-Kähler structures
Non-integrable geometries
Skew-symmetric torsion

ABSTRACT

We consider the question: Can the isotropy representation of an irreducible pseudo-Riemannian symmetric space be realized as a conformal holonomy group? Using recent results by Čap, Gover and Hammerl, we study the representations of $\mathrm{SO}(2,1)$, $\mathrm{PSU}(2,1)$ and $\mathrm{PSp}(2,1)$ as isotropy groups of irreducible symmetric spaces of signature $(3,2)$, $(4,4)$ and $(6,8)$, respectively, describing the geometry induced by a conformal holonomy reduction to the corresponding subgroups. In the case of $\mathrm{SO}(2,1)$, we show that conformal manifolds with such a holonomy reduction are always locally conformally flat and hence this group cannot be a conformal holonomy group. This result completes the classification of irreducible conformal holonomy groups in Lorentzian signature. In the case of $\mathrm{PSU}(2,1)$, we show that conformal manifolds of signature $(3,3)$ with this holonomy reduction carry, on an open dense subset, a canonical nearly para-Kähler metric with positive Einstein constant. For $\mathrm{PSp}(2,1)$ we also show that there is an open dense subset endowed with a canonical Einstein metric in the conformal class. As a result, after restricting to an open dense subset, the conformal holonomy must be a proper subgroup of $\mathrm{PSU}(2,1)$ or of $\mathrm{PSp}(2,1)$, respectively. These are special cases of an interesting relationship between a class of special conformal holonomy groups, and non-integrable geometries with skew symmetric, parallel torsion, which we also explore. Finally, using a recent result of Graham and Willse we prove the following general non-existence result: for a real-analytic, odd-dimensional conformal manifold, the conformal holonomy group can never be given by the isotropy representation of an irreducible pseudo-Riemannian symmetric space unless the isotropy group is $\mathrm{SO}^0(p+1,q+1)$.

☆ The authors acknowledge the support and hospitality by the International Erwin Schrödinger Institute in Vienna: Here the project was commenced during the workshop on "Cartan Connections, Geometry of Homogeneous Spaces and Dynamics", and the results where first presented at the workshop on "The Interaction of Geometry and Representation Theory". The last author was supported by the Australian Research Council through the fellowship FT110100429 and the grant DP120104582.

* Corresponding author.
E-mail addresses: jesse.alt@wits.ac.za (J. Alt), antonio.discala@polito.it (A.J. Di Scala), thomas.leistner@adelaide.edu.au (T. Leistner).

0926-2245/$ – see front matter
http://dx.doi.org/10.1016/j.difgeo.2013.10.012

1. Introduction and statement of results

A basic problem in differential geometry is to understand the holonomy representations of connections associated to geometric structures, e.g. to classify the holonomy representations which can be geometrically realized via a canonical connection. Given a linear connection on a principle fibre bundle, or equivalently, a covariant derivative ∇ on a vector bundle $\mathcal{T}$ over a manifold M, the holonomy group $\mathrm{Hol}_p(\mathcal{T}, \nabla)$ at a point $p \in M$ is defined as the group of parallel transports, with respect to ∇, along loops in M starting and ending at p. The holonomy group is a subgroup of $\mathrm{GL}(\mathcal{T}_p)$ that inherits a Lie group structure from its connected component, the *restricted holonomy group*. The restricted holonomy is obtained by restricting the definition to contractible loops, and hence, both groups are the same for simply connected manifolds.

The most well-known case is of course the classification of Riemannian holonomies, accomplished in the 20th century as a result of work by many mathematicians, including most notably Berger [10], Alekseevsky, Calabi, Yau, Bryant and Joyce (see [12,14,39] and references therein). For the classification of Lorentzian holonomy groups, see the survey [32] and references therein. Generalizations in various directions have been studied. For example, the classification of irreducible holonomy representations of torsion-free affine connections was completed by Merkulov and Schwachhöfer in [50].[1]

One possible generalization is to consider not a fixed pseudo-Riemannian metric, but only its conformal class, and study the holonomy associated to this geometric structure. When only a conformal equivalence class of pseudo-Riemannian metrics is fixed, the most natural connection to consider is not a principal connection but a *Cartan connection*. Given a Lie group G with Lie algebra $\mathfrak{g}$, a closed subgroup P and a principle P-bundle $\mathcal{G}$, a Cartan connection ω is a P-equivariant one-form on $\mathcal{G}$, that recovers fundamental vector fields and, in contrast to a principle fibre bundle connection, provides a global parallelism between $\mathcal{G}$ and $\mathfrak{g}$ (see the definition in Section 2). Hence, it does *not* define a horizontal distribution on $\mathcal{G}$, a fact which makes the notion of holonomy more involved. However, extending the bundle $\mathcal{G}$ to the G-bundle $\widehat{\mathcal{G}} := \mathcal{G} \times_P G$, the Cartan connection ω induces a principle fiber bundle connection $\widehat{\omega}$ on $\widehat{\mathcal{G}}$, and one can define the holonomy group of the Cartan connection ω as the usual holonomy group of $\widehat{\omega}$. There is a notion of holonomy for the Cartan connection that does not make use of this extension [57, Section 5.4], but one can prove [8, Proposition 1] that both have the same connected component. This is one of the reasons why we restrict ourselves to the study of the restricted holonomy group of Cartan connections — in the above sense as holonomy of $\widehat{\omega}$. In the following, when we use the word holonomy we will always refer to the restricted holonomy group. The other reason is that our approach is based on the Lie algebra of the holonomy group, the *holonomy algebra*, which can only describe the connected component of the full holonomy. We also use the notion of *holonomy representation*, which refers to the restricted holonomy group, or its Lie algebra, and its representation on the fiber $\mathcal{T}_p$ of the vector bundle on which the covariant derivative ∇ is defined.

The present work touches on the problem of classifying the representations which are realizable as the holonomy group of the canonical Cartan connection in conformal geometry, i.e., as "conformal holonomy groups". This means we study the holonomy representations of the canonical (normal) Cartan connection induced by a conformal manifold of dimension at least 3. (Definitions and some other relevant background material are reviewed in Sections 2.1, 2.2, 2.3.) In particular, note that this means that a conformal manifold $(M, [g])$ of signature (p, q) has conformal holonomy $\mathrm{Hol}(M, [g])$ given as a subgroup of $\mathrm{O}(p+1, q+1)$, and thus the basic holonomy representation is on the space $\mathbb{R}^{p+1,q+1} \simeq \mathcal{T}_p$.

The study of conformal holonomy has attracted considerable interest in recent years. The first fundamental observation that was made is that the conformal holonomy is contained in the stabilizer in $\mathrm{O}(p+1, q+1)$ of a line if and only if there exists an Einstein metric in the conformal class, where the Einstein metric might only be defined on an open dense subset [47,33,43]. As a result, we know that if the conformal class $[g]$ contains an Einstein metric, then the conformal holonomy representation preserves a line in $\mathbb{R}^{p+1,q+1}$, so it

[1] As the referee pointed out to us, some representations missing from this classification were later noticed in [15].