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Submaximal metric projective and metric affine structures

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ABSTRACT

We prove that the next possible dimension after the maximal n^2+2n for the Lie algebra of local projective symmetries of a metric on a manifold of dimension $n>1$ is n^2-3n+5 if the signature is Riemannian or $n=2$, n^2-3n+6 if the signature is Lorentzian and $n>2$, and n^2-3n+8 elsewise. We also prove that the Lie algebra of local affine symmetries of a metric has the same submaximal dimensions (after the maximal n^2+n) unless the signature is Riemannian and $n=3,4$, in which case the submaximal dimension is n^2-3n+6.

Introduction

Consider a linear torsion-free connection $\Gamma=(\Gamma^i_{jk})$ on a smooth connected manifold M^n of dimension $n\geqslant 2$. A vector field v is called a *projective symmetry*, or a *projective vector field*, if its local flow sends geodesics (considered as unparameterized curves) to geodesics. Since S. Lie [16] it is known that projective vector fields form a Lie algebra, which we denote by $\mathfrak{p}(\Gamma)$. A vector field v is called an *affine symmetry*, or an *affine vector field*, if its local flow preserves Γ; affine vector fields also form a Lie algebra $\mathfrak{a}(\Gamma)$, which we call *affine algebra*. Obviously $\mathfrak{a}(\Gamma)\subseteq\mathfrak{p}(\Gamma)$ is a subalgebra.

It follows from E. Cartan [2] that $\dim(\mathfrak{p}(\Gamma))\leqslant n^2+2n$ and a connection with the maximal dimension of the projective algebra is *projectively flat*, i.e. in a certain local coordinate system the geodesics are straight lines. I. Egorov [4] proved that the next possible dimension of $\mathfrak{p}(\Gamma)$, the so-called *submaximal* dimension (this is the maximal dimension among all non-flat structures), is n^2-2n+5 for $n>2$. For $n=2$, it was known since S. Lie [16] and A. Tresse [25] that the submaximal dimension is 3.

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However, for $n > 2$, the projective structures realizing this dimension are non-metric, in the sense there exists no (local) metric such that its Levi-Civita connection has $\dim(\mathfrak{p}) = n^2 - 2n + 5$. This observation follows for example from [7, (3.5)], which can be viewed as a system of linear equations on the components of the metric g, whose coefficients come from the components of the projective Weyl tensor W. By Egorov [4], in a certain coordinate system the connection with the submaximal dimension of the projective algebra has only two nonzero term $\Gamma^1_{23} = \Gamma^1_{32} = x_2$. Direct calculation shows that the only nonvanishing components of the projective Weyl tensor are $W^1_{232} = 1 = -W^1_{322}$; substitution of this into [7, (3.5)] yields a system of linear equations such that any solution g is a degenerate symmetric tensor.

Non-metrizability of the Egorov's submaximal projective structure was obtained independently (and by another method) by S. Casey and M. Dunajski. In fact, it follows instantly from our first main result (below δ^2_n is the Kronecker symbol, i.e. 1 for $n = 2$ and 0 else):

Theorem 1. *Let Γ be the Levi-Civita connection of a metric g on M^n. Assume that Γ is not projectively flat at least at one point (i.e. g is not a metric of constant sectional curvature). Then the maximal possible dimension of the symmetry algebra $\mathfrak{p}(g) = \mathfrak{p}(\Gamma)$ is equal to*

- $n^2 - 3n + 5$, *if g has Riemannian signature,*[1]
- $n^2 - 3n + 6 - \delta^2_n$, *if g has Lorentzian signature,*
- $n^2 - 3n + 8$, *if g has the general signature.*

The bound for the general signature was obtained by Mikes [21]. Our approach however differs from his.

Notice that in the global setting, i.e. if we replace the projective algebra by a projective group, the sub-maximal projective connection is metric. The reason is that there are locally-projectively-flat manifolds whose projective group actually has dimension $n^2 + n$ [27] (this is the global submaximal bound).

If M is closed and the metric g is Riemannian of non-constant sectional curvature, then the sub-maximal bound is $\binom{n}{2} + 1$ for all $n \neq 4$; for $n = 4$ this dimension is $\binom{n}{2} + 2 = 8$. Indeed by [19,20], on closed Riemannian manifolds of nonconstant sectional curvature, the projective group acts by isometries, so the claim follows from [28,5,11]. The corresponding models are precisely $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ for $n \neq 4$, possibly quotient by a finite group, and $\mathbb{C}P^2$ for $n = 4$.

The problem of determining the dimension gaps (lacunes in the terminology of the Russian geometry school) between the maximal and sub-maximal structures is classical, see the discussion in [14].

Let us now discuss an analogous question for the affine algebra. The maximal dimension of the space of affine symmetries of an affine connection is classically known to be $n^2 + n$. The submaximal dimension is n^2 and this was also found by I. Egorov [6]. The corresponding connections are projectively flat, and for projectively non-flat connections the affine algebra has maximal dimension $n^2 - 2n + 5$, $n > 2$ [4] (it equals 3 for $n = 2$). Again, all these submaximal connections are non-metric.

Our second main result concerns submaximal dimensions of the affine algebras $\mathfrak{a}(\Gamma)$ of Levi-Civita connections Γ.

Theorem 2. *Non-flat metrics g on a manifold M^n have maximal dimension of the affine algebra $\mathfrak{a}(g) = \mathfrak{a}(\Gamma)$ equal to*

- $n^2 - 3n + 5 + \delta^3_n + \delta^4_n$, *if g has Riemannian signature,*
- $n^2 - 3n + 6 - \delta^2_n$, *if g has Lorentzian signature,*
- $n^2 - 3n + 8$, *if g has the general signature.*

[1] Within this paper we consider the metrics up to multiplication by ± 1 (since multiplication by a nonzero constant does not change the projective and affine algebras). In particular both signatures $(+, -, \dots, -)$ and $(-, +, \dots, +)$ are Lorentzian for us, and we view positively and negatively definite metrics as Riemannian.