Differential Geometry and its Applications 33 (2014) 81–111

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Highly symmetric 2-plane fields on 5-manifolds and 5-dimensional Heisenberg group holonomy ☆

Travis Willse

Mathematical Sciences Institute, John Dedman Building 27, The Australian National University, ACT 0200, Australia

ARTICLE INFO

Article history:
Available online 22 November 2013
Communicated by A.R. Gover

MSC:
53A30
53B15
53C15
53C26
53C29

Keywords:
Conformal geometry
Fefferman–Graham ambient metric
Generic distributions
Special holonomy

ABSTRACT

Nurowski showed that any generic 2-plane field D on a 5-manifold M determines a natural conformal structure c_D on M; these conformal structures are exactly those (on oriented M) whose normal conformal holonomy is contained in the (split, real) simple Lie group G_2. Graham and Willse showed that for real-analytic D the same holds for the holonomy of the real-analytic Fefferman–Graham ambient metric of c_D, and that both holonomy groups are equal to G_2 for almost all D. We investigate here independently interesting 2-plane fields for which the associated holonomy groups are a proper subgroup of G_2.

Cartan solved the local equivalence problem for 2-plane fields D and constructed the fundamental curvature tensor A for these objects. He furthermore claimed to describe locally all D whose infinitesimal symmetry algebra has rank at least 6 and gave a local quasi-normal form, depending on a single function of one variable, for those that furthermore satisfy a natural degeneracy condition on A, but Doubrov and Govorov recently rediscovered a counterexample to Cartan's claim. We show that for all D given by Cartan's alleged quasi-normal form, the conformal structures c_D induced via Nurowski's construction are almost Einstein, that we can write their ambient metrics explicitly, and that the holonomy groups associated to c_D are always the 5-dimensional Heisenberg group, which here acts indecomposably but not irreducibly. (Not all of these properties hold, however, for Doubrov and Govorov's counterexample.) We also show that the similar results hold for the related class of 2-plane fields defined on suitable jet spaces by ordinary differential equations $z'(x) = F(y''(x))$ satisfying a simple genericity condition.

1. Introduction

In a well-known but technically demanding 1910 paper, [6], Cartan solved the local equivalence problem for what in modern geometric language are called 2-plane fields on 5-manifolds, and the most interesting such fields are those that satisfy a simple genericity condition. This class is the lowest-dimensional example of k-plane fields on n-manifolds that admit nontrivial local invariants, but already the geometry of these

☆ This article is dedicated to Mike Eastwood on the occasion of his 60th birthday.

E-mail address: travis.willse@anu.edu.au.

0926-2245/$ – see front matter
http://dx.doi.org/10.1016/j.difgeo.2013.10.010

fields is surprisingly rich and furthermore enjoys close connections with some exceptional geometric objects, including the algebra of the split octonions and the exceptional Lie group G_2.

One of the most striking realizations of these connections was described by Nurowski [27,28] and Leistner and Nurowski [26], whose work exploits the geometry of generic 2-plane fields D on 5-manifolds M to produce metrics of holonomy equal to G_2 (here and henceforth, G_2 denotes the split real form of the exceptional Lie group). They produce candidate metrics of this kind by concatenating two constructions: First, Nurowski exploited Cartan's solution of the local equivalence problem for these 2-plane fields to show that any such field D induces a canonical conformal structure c_D of signature $(2,3)$ on the underlying manifold [27]. Second, the Fefferman–Graham ambient construction associates to any conformal structure (M,c) of signature (p,q) an essentially unique metric $\widetilde{g}$ of signature $(p+1,q+1)$ on a suitable open subset $\widetilde{M} \subseteq \mathbb{R}_+ \times M \times \mathbb{R}$, though for most c the metric $\widetilde{g}$ cannot be identified explicitly [14]. Applying this latter construction to a conformal structure c_D produces a pseudo-Riemannian metric of signature $(3,4)$, and Leistner and Nurowski produced an explicit family of 2-plane fields D parametrized by $\mathbb{R}^8$ and found corresponding (polynomial) ambient metrics $\widetilde{g}_D$ of c_D. By giving explicitly a certain object parallel with respect to $\widetilde{g}_D$—namely, a 3-form of a certain algebraic type—they showed that the holonomy groups $\mathrm{Hol}(\widetilde{g}_D)$ of the metrics in this family all are contained in the stabilizer in $\mathrm{SO}(3,4)$ of the 3-form, which turns out to be G_2, and moreover that for an explicit, dense open subset of parameter values, $\mathrm{Hol}(\widetilde{g}_D) = \mathrm{G}_2$ [26]. This is interesting in part because there are relatively few examples of metrics with this holonomy group. Later, Graham and Willse showed that for all real-analytic D on oriented 5-manifolds, there is an ambient metric $\widetilde{g}_D$ such that $\mathrm{Hol}(\widetilde{g}_D) \leqslant \mathrm{G}_2$ and that, in a suitable sense, equality holds generically [16].

In this article we give an explicit infinite-dimensional family of 2-plane fields D and corresponding explicit ambient metrics $\widetilde{g}_D$ for which the containment of holonomy in G_2 is proper. The 2-plane fields in this family satisfy two strong invariant criteria (but, pace Cartan's claims, the family is not characterized by these conditions). First, Cartan described the fundamental curvature quantity of 2-plane fields D, which we may interpret as a tensor field $A \in \Gamma(\odot^4 D^*)$ [6]. If we complexify A, then for each $u \in M$, we may regard the roots of $A_u \otimes \mathbb{C} \in \odot^4 D_u^* \otimes \mathbb{C}$ as elements of the complex projective line $\mathbb{P}(D_u \otimes \mathbb{C})$, and if $A_u \neq 0$, we call the partition of 4 given by the root multiplicities the root type of D at u; for example, we say that D has root type [4] at u if $A_u \otimes \mathbb{C}$ is nonzero and has a quadruple root, or equivalently, if the line it spans is contained in the rational normal curve in $\odot^4 D_u^* \otimes \mathbb{C}$. Second, the (infinitesimal) symmetry algebra of D is the Lie algebra $\mathfrak{aut}(D)$ of vector fields that preserve D. Cartan claimed that one can locally encode any 2-plane field D such that (A) D has constant root type [4] (that is, root type [4] at every $u \in M$), and (B) $\dim \mathfrak{aut}(D) \geqslant 6$, in a principal bundle $E \to M$ with 2-dimensional structure group and a coframe (13) on E that depends only a single function I, but Doubrov and Govorov have recently produced an interesting counterexample to this claim. For any smooth function I the structure equations of this coframe determines a 2-plane field D_I, and we produce explicit ambient metrics $\widetilde{g}_I$ (15) on spaces $\widetilde{M}_I$ for the conformal structures $c_I := c_{D_I}$ they induce. These conformal structures all enjoy additional special structures, including (exactly) a 2-dimensional vector space of almost Einstein scales (in fact, almost Ricci-flat scales). Each of these scales in turn corresponds to a parallel null vector field on $\widetilde{M}_I$, and so $\mathrm{Hol}(\widetilde{g}_I)$ must be contained inside the common stabilizer of those vector fields; we show that $\mathrm{Hol}(\widetilde{g}_I)$ is actually this full group, which roughly indicates that the only objects parallel with respect to $\widetilde{g}_I$ are those arising from the parallel G_2-structure and the described null vector fields.

We also compute for the conformal structures c_I a closely related notion of holonomy that has recently enjoyed heightened attention [25,19,9]. One can encode any n-dimensional conformal manifold (M,c) in a rank-$(n+2)$ bundle $\mathcal{T} \to M$ called the tractor bundle, together with some auxiliary data including a canonical (normal) conformal connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$; we call the holonomy $\mathrm{Hol}(\nabla^{\mathcal{T}})$ of this connection the (normal) conformal holonomy of c.

Partly via explicit computation we prove the following: