Differential Geometry and its Applications 33 (2014) 144–161

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Finding solutions of parabolic Monge–Ampère equations by using the geometry of sections of the contact distribution

Dmitri Alekseevsky [a,1], Ricardo Alonso-Blanco [b], Gianni Manno [c,*,2], Fabrizio Pugliese [d]

[a] *A.A. Kharkevich Institute for Information Transmission Problems, Bol'shoi Karetnyi 19, 127994 Moscow, Russia*
[b] *Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain*
[c] *Mathematisches Institut, Friedrich-Schiller-Universität Jena, 07737 Jena, Germany*
[d] *Dipartimento di Matematica, Università di Salerno, via Ponte don Melillo, 84084 Fisciano, Italy*

ARTICLE INFO

Article history:
Available online 18 November 2013
Communicated by R. Graham

MSC:
35K96
53B05
53D10
58A20
58A30

Keywords:
Parabolic Monge–Ampère equations
Characteristic distribution
Construction of solutions

ABSTRACT

In a series of papers we have described normal forms of parabolic Monge–Ampère equations (PMAEs) by means of their characteristic distribution. In particular, PMAEs with two independent variables are associated with Lagrangian (or Legendrian) subdistributions of the contact distribution of a 5-dimensional contact manifold. The geometry of sections of the contact distribution allowed us to get the aforementioned normal forms. In the present work, for a distinguished class of PMAEs, we will construct 3-parametric families of solutions starting from particular sections of the characteristic distribution. We will illustrate the method by several concrete computations. Moreover, we will see, for some linear PMAEs, how to construct a recursive process for obtaining new solutions. At the end, after showing that some classical equations on affine connected 3-dimensional manifolds are PMAEs, we will apply the integration method to some particular examples.

1. Introduction

1.1. Notation and conventions

Throughout this paper, everything is supposed to be C^∞ and local. For simplicity, when X is a vector field and $\mathcal{P}$ is a distribution on the same manifold, we write "$X \in \mathcal{P}$" to indicate that X is a smooth (local) section of tangent subbundle $\mathcal{P}$. We will use $X(T)$ to denote the Lie derivative of a tensor T along X. We

* Corresponding author.
E-mail addresses: D.Aleksee@ed.ac.uk (D. Alekseevsky), ricardo@usal.es (R. Alonso-Blanco), gianni.manno@uni-jena.de (G. Manno), fpugliese@unisa.it (F. Pugliese).
[1] Present address: Masarik University, Department of Mathematics, Kotlarska 2, 61137 Brno, Czech Republic.
[2] INdAM-COFUND Marie Curie Fellow.

0926-2245/$ – see front matter
http://dx.doi.org/10.1016/j.difgeo.2013.10.015

denote by $\langle v_1, v_2\rangle$ the linear span of vectors v_1, v_2. Finally, first and second order jet coordinates will be indifferently denoted either by $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ or p, q, r, s, t, respectively.

1.2. Description of the main results

In this paper we present a method to obtain families of solutions of *parabolic* Monge–Ampère equations (PMAEs). Such method is based on the geometrical approach to PMAEs developed in several previous papers (see [2–5]).

MAEs are second order PDEs of the form

$$N\left(z_{xx}z_{yy} - z_{xy}^2\right) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0, \tag{1}$$

in the unknown function $z = z(x, y)$, with coefficients A, B, C, D and N depending on x, y, z, z_x, z_y. As it is well known, any contact transformation maps a MAE into another one. As we said, in what follows we will be concerned with PMAEs, which form the following subclass of (1):

$$N\left(z_{xx}z_{yy} - z_{xy}^2\right) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0, \qquad B^2 - 4AC + 4ND = 0. \tag{2}$$

Geometrically, this means that characteristic directions at any point of the contact manifold $M = \{(x, y, z, z_x, z_y)\}$ fill a Lagrangian distribution $\mathcal{D} \subset \mathcal{C}$, where $\mathcal{C} = \operatorname{Ker}\theta$ with $\theta = dz - z_x\,dx - z_y\,dy$ is the contact distribution of M. We call $\mathcal{D}$ the *characteristic distribution* of the PMAE. As $\mathcal{D}$, generally, is not integrable, it is necessary to consider also its derived flag

$$\mathcal{D} \subset \mathcal{D}' = \mathcal{D} + [\mathcal{D}, \mathcal{D}] \subset \mathcal{D}'' = \mathcal{D}' + \left[\mathcal{D}', \mathcal{D}'\right]. \tag{3}$$

Generically, $\mathcal{D}'' = TM$. Symmetry properties of (3) allow to obtain important classification results on PMAEs in a simple and straightforward way. In fact, such a study is based, to a large extent, on the geometry of sections of the contact distribution $\mathcal{C}$. Such sections are not all contact-equivalent: there exists a simple invariant, called *type*, which is an integer number belonging to $\{2, 3, 4\}$ that measures the degree of "genericity" of a section X of $\mathcal{C}$: the higher the type, the less symmetric X is with respect to $\mathcal{C}$. More precisely, X is of type $2, 3$ or 4 if it is contained in many, one or no integrable 2-dimensional subdistribution of $\mathcal{C}$, respectively. The above vector fields are linked to the existence of various kinds of intermediate integrals (classical, non-holonomic, complete) of a PMAE as follows. Roughly speaking, a complete integral of (2) is any 3-parametric family of solutions: its existence is equivalent to the existence of a *generalized intermediate integral*, i.e. a section of the characteristic distribution $\mathcal{D} \subset \mathcal{C}$ of type less than 4. This is a generalization of both the classical [7] and the non-holonomic [10] notion of intermediate integral; in fact, a classical intermediate integral $f \in C^\infty(M)$ of (2) can be identified with a *Hamiltonian field* X_f (a special kind of section of the contact distribution of type 2) contained in $\mathcal{D}$, whereas a non-holonomic intermediate integral is any section of $\mathcal{D}$ of type 2 (see also [14, p. 112]).

The existence of intermediate integrals of Eq. (2) is strictly linked to integrability properties of the derived flag (3). In particular, we will focus on the following case: when $\mathcal{D}''$ is 4-dimensional and integrable, the associated PMAE is contact-equivalent to

$$z_{yy} = b(x, y, z, z_x, z_y), \qquad \partial_{z_x}(b) \neq 0. \tag{4}$$

The characteristic distribution of (4) is

$$\mathcal{D} = \langle \partial_p, \widehat{\partial}_y + b\partial_q\rangle, \quad \widehat{\partial}_y := \partial_y + q\partial_z.$$

The method for finding complete integrals of Eq. (4) consists of 4 steps: