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Finsler spaces whose geodesics are orbits ☆

Zaili Yan [a], Shaoqiang Deng [b,*]

[a] School of Mathematical Sciences, Nankai University, Tianjin 300071, People's Republic of China
[b] School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People's Republic of China

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ABSTRACT

In this paper, we study Finsler spaces whose geodesics are the orbits of one-parameter subgroups of the group of isometries (abbreviated as Finsler g.o. spaces). We first generalize some geometric results on Riemannian g.o. spaces to the Finslerian setting. Then we show that a Finsler g.o. nilmanifold is at most two step nilpotent and construct some examples of g.o. spaces which are neither Berwaldian nor weakly symmetric. Further, we give a sufficient and necessary condition for a Randers space to be a g.o. space. Finally, we show that every Clifford–Wolf homogeneous Finsler space is a Finsler g.o. space.

1. Introduction

A geodesic $\gamma : \mathbb{R} \to M$ on a Riemannian manifold (M, Q) is called homogeneous if there is a one-parameter group of isometries $\phi : \mathbb{R}\times M \to M$ such that $\gamma(t) = \phi(t, \gamma(0))$, $t \in \mathbb{R}$. The notion of a homogeneous geodesic plays a fundamental role in the theory of homogeneous Riemannian manifold, especially in the study of g.o. spaces (i.e., a Riemannian manifold whose geodesics are all homogeneous). It is well known that any naturally reductive Riemannian manifold is a g.o. space. Several decades ago, it was generally believed that the notion of a Riemannian g.o. manifold is just an equivalent description of the property of being naturally reductive [3]. The first counter-example of a g.o. space which is in no way naturally reductive was found by Kaplan [27]. This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so-called H-type groups. Kaplan's paper leads to an extensive study of g.o. spaces. For example, Riehm [33] classified all g.o. nilmanifolds of H-type groups. Later, Kowalski and Vanhecke [29] proved that, up to

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* Corresponding author.
E-mail addresses: yanzaili@mail.nankai.edu.cn (Z. Yan), dengsq@nankai.edu.cn (S. Deng).

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dimension 5, every g.o. space is, or can be made to be, naturally reductive. Then in [24], Gordon reduced the classification of g.o. spaces to three special cases: (1) g.o. nilmanifolds (i.e., a nilpotent Lie group with a left invariant Riemannian metric), (2) compact g.o. spaces, and (3) a Riemannian manifold admitting a transitive non-compact semisimple Lie group of isometries. Gordan also gave a rather detailed description of g.o. nilmanifolds. Recently, Dušek, Kowalski and Nikčević [22] constructed some examples of g.o. spaces in dimension 7 whose full groups of isometries are not semisimple. However, the general problem of the classification of homogeneous Riemannian g.o. manifolds remains open. Up to now, there are only partial results concerning the compact case; see the works of Alekseevsky and Arvanitoyeorgos [1], Alekseevsky and Nikonorov [2].

In this paper, we will study g.o. spaces in the Finslerian setting. Since the full group of isometries of a Finsler space is a Lie group [14], we can define a Finsler g.o. space in exactly the same way as in the Riemannian case, namely, a Finsler g.o. space is a space such that every geodesic is the orbit of a one-parameter group of isometries. We remark here that the notion of a Finsler g.o. space not only is just an analogue of Riemannian g.o. spaces, but also has its own merits in Finsler geometry. For example, every Finsler g.o. space has vanishing S-curvature, thus the Bishop–Gromove volume comparison theorem is true for any Finsler g.o. space [36]. Up to now, the only known examples of Finsler g.o. spaces are the Riemannian ones and weakly symmetric Finsler spaces [12]. It would be interesting to find more new examples of Finsler g.o. spaces, particularly those g.o. spaces which are neither Berwaldian nor weakly symmetric.

The arrangement of this paper is as the following. In Section 2, we present some preliminaries on Finsler geometry. In particular, we introduce the Chern connection and the notion of S-curvature of a Finsler space. Section 3 is devoted to studying the general geometric properties of Finsler g.o. spaces. In Section 4, we study Berwald g.o. spaces. In Section 5, we study Finsler g.o. nilmanifolds. In particular, we show that a Finsler g.o. nilmanifold is at most two step nilpotent and present some examples of Finsler g.o. spaces which are neither Berwaldian nor weakly symmetric. In Section 6, we consider Randers g.o. spaces. We present a sufficient and necessary condition for a Randers space to be a g.o. space. Finally, in Section 7, we study Clifford–Wolf homogeneous Finsler spaces, and prove that any such space is a g.o. space.

2. Preliminaries

In this section, we recall briefly some known facts about Finsler spaces. For details, see [5,10,13,38].

Definition 2.1. Let V be a n-dimensional real vector space and F a real function on V which is smooth on $V\backslash\{0\}$ and satisfies the following conditions:

(1) $F(u) \geq 0$, $\forall u \in V$;
(2) $F(\lambda u) = \lambda F(u)$, $\forall \lambda > 0$;
(3) For any basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of V, write $F(y) = F(y^1, y^2, \ldots, y^n)$ for $y = y^i\varepsilon_i$. Then the Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^iy^j}\right)$$

is positive-definite at any point of $V\backslash\{0\}$.

Then F is called a Minkowski norm on V. In this case, (V, F) is called a Minkowski space.

For any Minkowski norm F on real vector space V, one defines

$$C_{ijk} = \frac{1}{4}\left[F^2\right]_{y^iy^jy^k}.$$