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Special bi-invariant linear connections on Lie groups and finite dimensional Poisson structures ☆

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ABSTRACT

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. We denote by ∇^0 the torsion free bi-invariant linear connection on G given by $\nabla^0_X Y = \frac{1}{2}[X,Y]$, for any left invariant vector fields X, Y. A Poisson structure on $\mathfrak{g}$ is a commutative and associative product on $\mathfrak{g}$ for which ad_u is a derivation, for any $u \in \mathfrak{g}$. A torsion free bi-invariant linear connections on G which have the same curvature as ∇^0 are called special. We show that there is a bijection between the space of special connections on G and the space of Poisson structures on $\mathfrak{g}$. We compute the holonomy Lie algebra of a special connection and we show that the Poisson structures associated to special connections which have the same holonomy Lie algebra as ∇^0 possess interesting properties. Finally, we study Poisson structures on a Lie algebra and we give a large class of examples which gives, of course, a large class of special connections.

1. Introduction

A Poisson algebra is both a Lie algebra and a commutative associative algebra which are compatible in a certain sense. Poisson algebras play important roles in many fields in mathematics and mathematical physics, such as the Poisson geometry, integrable systems, non-commutative (algebraic or differential) geometry, and so on. Finite dimensional Poisson algebras constitute an interesting topic in algebra and were studied by many authors (see for instance [13,19,21]).

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More precisely, a *Poisson algebra* is a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ endowed with a commutative and associative product $\circ$ such that, for any $u, v, w \in \mathfrak{g}$,

$$[u, v \circ w] = [u, v] \circ w + v \circ [u, w]. \tag{1}$$

An algebra $(\mathrm{A}, .)$ is called *Poisson admissible* if $(\mathrm{A}, [\ ,\ ], \circ)$ is a Poisson algebra, where

$$[u, v] = u.v - v.u \quad \text{and} \quad u \circ v = \frac{1}{2}(u.v + v.u). \tag{2}$$

This paper aims to give some new insights on finite dimensional Poisson algebras based on an interesting geometric interpretation of these structures when the field is either $\mathbb{R}$ or $\mathbb{C}$ (see Theorem 2.1). Let us present briefly this geometric interpretation.

Let G be a Lie group with $\mathfrak{g} = T_eG$ its Lie algebra. The linear connection ∇^0 given by $\nabla^0_X Y = \frac{1}{2}[X, Y]$, where X, Y are left invariant vector fields, is torsion free, bi-invariant, complete and its curvature K^0 is given by $K^0(X, Y) = -\frac{1}{4}\mathrm{ad}_{[X,Y]}$. Moreover, $\nabla^0 K^0 = 0$ and the holonomy Lie algebra of ∇^0 at e is $\mathfrak{h}^0 = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$. The main fact (see Section 2) is that there is a bijection between the set of Poisson structures on $\mathfrak{g}$ and the space of bi-invariant torsion free linear connections on G which have the same curvature as ∇^0. We call such connections *special*. Moreover, we show that any special connection is semi-symmetric, i.e., its curvature tensor K satisfies $K.K = 0$ (see Proposition 2.1). In general, the holonomy Lie algebra $\mathfrak{h}$ of a bi-invariant linear connection is difficult to compute, however, we show that, for a special connection, $\mathfrak{h}$ contains $\mathfrak{h}^0$ and can be easily computed (see Lemma 2.2). A special connection whose holonomy Lie algebra coincides with $\mathfrak{h}^0$ will be called *strongly special*. So, according to the bijection above, to any real Poisson algebra corresponds a unique special connection on any associated Lie group. Poisson algebras whose corresponding special connection is strongly special are particularly interesting. We call such Poisson algebras *strong*. A Poisson algebra whose corresponding special connection is parallel is called *parallel*. With this interpretation in mind, we devote Section 3 to the study of the general properties of Poisson algebras and Poisson admissible algebras and we give some general methods to build new Poisson algebras from old ones (see Theorem 3.2). We show that any symmetric Leibniz algebra is a strong Poisson admissible algebra and the curvature of the corresponding special connection is parallel (see Theorem 3.1). By using the geometric interpretation of Poisson structures, we get a large class of Lie groups which carry a bi-invariant connection ∇ (different from ∇^0) which has the same curvature and the same linear holonomy as ∇^0 and moreover the curvature of ∇ is parallel. We get hence interesting examples of connections with parallel torsion and curvature. Such connections were studied by Nomizu [20]. Recall that symmetric Leibniz algebras constitute a subclass of Leibniz algebras introduced by Loday in [18]. At the end of Section 3, we show that there is no non-trivial Poisson structure on a semi-simple Lie algebra (see Theorem 3.3). This result generalizes a result by [13]. In Section 4, we show that an associative algebra is Poisson admissible if and only if the underline Lie algebra is 2-nilpotent and we give a description of associative Poisson admissible algebras which permit to build many examples. Section 5 is devoted to the study of symplectic Poisson algebras. It is well-known that if $(\mathfrak{g}, \omega)$ is a symplectic Lie algebra there is a product α^{a} on $\mathfrak{g}$ which is Lie-admissible and left symmetric. When the Lie algebra is real, α^{a} defines a left invariant flat torsion free linear connection ∇^{a} on any associated Lie group G. By using the general method to build a torsion free symplectic connection from any torsion free connection introduced in [4], we get from ∇^{a} a left invariant torsion free connection ∇^{s} for which the left invariant symplectic form associated to ω is parallel. To our knowledge this connection has never been considered before. From ∇^{s} we get a product α^{s} on $\mathfrak{g}$. We show that $(\mathfrak{g}, \alpha^{\mathrm{a}})$ is Poisson admissible iff $(\mathfrak{g}, \alpha^{\mathrm{s}})$ is Poisson admissible and this is equivalent to $\mathfrak{g}$ is 2-nilpotent Lie algebra and $[\mathrm{ad}_u, \mathrm{ad}^*_v] = 0$ for any $u, v \in \mathfrak{g}$ where ad^*_u is the adjoint of ad_u with respect to ω. A symplectic Lie algebra satisfying these conditions is called *symplectic Poisson algebra*. We show that the symplectic double extension process introduced in [10] permits the construction of all symplectic Poisson algebras. Lie groups whose Lie algebras are symplectic Poisson