Differential Geometry and its Applications 35 (2014) 2–18

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

New homogeneous Einstein metrics on Stiefel manifolds ☆

Andreas Arvanitoyeorgos [a,*], Yusuke Sakane [b], Marina Statha [a]

[a] University of Patras, Department of Mathematics, GR-26500 Rion, Greece
[b] Osaka University, Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Toyonaka, Osaka 560-0043, Japan

A R T I C L E I N F O

Article history:
Available online 5 February 2014
Communicated by O. Kowalski

MSC:
primary 53C25
secondary 53C30, 13P10, 65H10, 68W30

Keywords:
Homogeneous space
Einstein metric
Stiefel manifold
Isotropy representation
Gröbner basis

A B S T R A C T

We consider invariant Einstein metrics on the Stiefel manifold $V_q\mathbb{R}^n$ of all orthonormal q-frames in $\mathbb{R}^n$. This manifold is diffeomorphic to the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-q)$ and its isotropy representation contains equivalent summands. We prove, by assuming additional symmetries, that $V_4\mathbb{R}^n$ $(n \geqslant 6)$ admits at least four $\mathrm{SO}(n)$-invariant Einstein metrics, two of which are Jensen's metrics and the other two are new metrics. Moreover, we prove that $V_5\mathbb{R}^7$ admits at least six invariant Einstein metrics, two of which are Jensen's metrics and the other four are new metrics.

1. Introduction

A Riemannian manifold (M,g) is called Einstein if it has constant Ricci curvature, i.e. $\mathrm{Ric}_g = \lambda \cdot g$ for some $\lambda \in \mathbb{R}$. A detailed exposition on Einstein manifolds can be found in [8,17] and [18]. General existence results are difficult to obtain and some methods are described in [9,10] and [19]. Also, in [4] the authors introduced a method for proving existence of homogeneous Einstein metrics by assuming additional symmetries.

In the present article we are interested in invariant Einstein metrics on homogeneous spaces G/H whose isotropy representation is decomposed into a sum of irreducible but possibly equivalent summands. By using the method of [4] we study homogeneous Einstein metrics on Stiefel manifolds.

The Stiefel manifolds are the homogeneous spaces $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ and the simplest case is the sphere $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$, which is an irreducible symmetric space, therefore it admits up to scale a unique invariant Einstein metric. Concerning Einstein metrics on other Stiefel manifolds we review the

☆ The second author was supported by JSPS KAKENHI Grant Number 25400071.
* Corresponding author.
E-mail addresses: arvanito@math.upatras.gr (A. Arvanitoyeorgos), sakane@math.sci.osaka-u.ac.jp (Y. Sakane), statha@master.math.upatras.gr (M. Statha).

0926-2245/$ – see front matter
http://dx.doi.org/10.1016/j.difgeo.2014.01.007

following: In [14] S. Kobayashi proved existence of an invariant Einstein metric on the unit tangent bundle $T_1S^n = \mathrm{SO}(n)/\mathrm{SO}(n-2)$. In [16] A. Sagle proved that the Stiefel manifolds $V_k\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-k)$ admit at least one homogeneous Einstein metric. For $k \geqslant 3$ G. Jensen in [12] found a second metric. If $n = 3$ the Lie group $\mathrm{SO}(3)$ admits a unique Einstein metric. For $n \geqslant 5$ A. Back and W.Y. Hsiang in [7] proved that $\mathrm{SO}(n)/\mathrm{SO}(n-2)$ admits exactly one homogeneous Einstein metric. The same result was obtained by M. Kerr in [13] by proving that the diagonal metrics are the only invariant metrics on $V_2\mathbb{R}^n$ (see also [3,1]). The Stiefel manifold $\mathrm{SO}(4)/\mathrm{SO}(2)$ admits exactly two invariant Einstein metrics ([2]). One is Jensen's metric and the other one is the product metric on $S^3 \times S^2$.

Finally, in [4] the first author, V.V. Dzhepko and Yu. G. Nikonorov proved that for $s > 1$ and $\ell > k \geqslant 3$ the Stiefel manifold $\mathrm{SO}(sk+\ell)/\mathrm{SO}(\ell)$ admits at least four $\mathrm{SO}(sk+\ell)$-invariant Einstein metrics, two of which are Jensen's metrics. The special case $\mathrm{SO}(2k+\ell)/\mathrm{SO}(\ell)$ admitting at least four $\mathrm{SO}(2k+\ell)$-invariant Einstein metrics was treated in [5]. Corresponding results for the quaternionic Stiefel manifolds $\mathrm{Sp}(sk+\ell)/\mathrm{Sp}(\ell)$ were obtained in [6]. In the present paper we prove that $\mathrm{SO}(n)/\mathrm{SO}(n-4)$ admits two more $\mathrm{SO}(n)$-invariant Einstein metrics and that $\mathrm{SO}(7)/\mathrm{SO}(2)$ admits four more $\mathrm{SO}(7)$-invariant Einstein metrics.

2. *G*-Invariant metrics with additional symmetries

We review a construction originally introduced in [4]. Let G be a compact Lie group and H a closed subgroup so that G acts almost effectively on G/H. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebra of G and H and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$ with respect to some $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. The orthogonal complement $\mathfrak{m}$ can be identified with the tangent space $T_o(G/H)$, where $o = eH$. Any G-invariant metric g of G/H corresponds to an $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$ and vice versa. For G semisimple the negative of the Killing form B of $\mathfrak{g}$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, therefore we can choose the above decomposition with the respect to $-B$.

The normalizer $N_G(H)$ of H in G acts on G/H by $(\alpha, gH) \mapsto g\alpha^{-1}H$. For a fixed $\alpha \in N_G(H)$ this action induces a G-equivariant diffeomorphism $\varphi_\alpha : G/H \to G/H$. Let K be a closed subgroup of G with $H \subset K \subset G$ such that $K = L' \times H'$, where L', H' are subgroups of G with $H = \{e_{L'}\} \times H'$. It is clear that $K \subset N_G(H)$. Consider the group $L = L' \times \{e_{H'}\}$. Then the group $\widetilde{G} = G \times L$ acts on G/H by $(\alpha, \beta)\cdot gH = \alpha g\beta^{-1}H$ and the isotropy $\widetilde{H}$ at eH is given as follows:

Lemma 2.1. *(See [4, Lemma 2.1].) The isotropy subgroup $\widetilde{H}$ is isomorphic to K.*

Hence, we have the diffeomorphisms

$$G/H \cong (G \times L)/K = (G \times L)/\bigl(L' \times H'\bigr) \cong \widetilde{G}/\widetilde{H}. \tag{1}$$

The elements of the set $\mathcal{M}^G$, of G-invariant metrics on G/H, are in 1–1 correspondence with $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$. We now consider $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$ (and not only $\mathrm{Ad}(H)$-invariant) with corresponding subset $\mathcal{M}^{G,K}$ of $\mathcal{M}^G$. Let $g \in \mathcal{M}^{G,K}$ and $\alpha \in K$. The diffeomorphism φ_α is an isometry of $(G/H, g)$. The action of $\widetilde{G}$ on $(G/H, g)$ is isometric, so any metric in $\mathcal{M}^{G,K}$ can be identified as a metric in $\mathcal{M}^{\widetilde{G}}$, the set of $\widetilde{G}$-invariant metrics on $\widetilde{G}/\widetilde{H}$, and vice versa. Therefore, we have $\mathcal{M}^{\widetilde{G}} \subset \mathcal{M}^G$.

We consider the isotropy representation $\chi : H \to \mathrm{Aut}(\mathfrak{m})$ of G/H, which is the restriction of the adjoint representation of H on $\mathfrak{m}$. We assume that χ decomposes into a direct sum of irreducible sub-representations

$$\chi \cong \chi_1 \oplus \cdots \oplus \chi_s,$$

giving rise to a decomposition