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Eigenvalues of the Tachibana operator which acts on differential forms

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A B S T R A C T

In the present paper we show spectral properties of a little-known natural Riemannian second-order differential operator acting on differential forms.

1. Introduction

1.1. Spectral geometry is the field of mathematics which deals with relationships between geometric structures of an n-dimensional Riemannian manifold (M, g) and the spectra of canonical differential operators. The field of spectral geometry is a vibrant and active one.

In the present paper we show spectral properties of a little-known natural Riemannian second-order differential operator acting on differential forms.

The paper is organized as follows. First, we consider the basis of the space of natural Riemannian (with respect to isometric diffeomorphisms of Riemannian manifolds) first-order differential operators on

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differential r-forms ($1 \leqslant r \leqslant n-1$) with values in the space of homogeneous tensors on (M, g). This basis consists of three operators $\{d, d^*, D\}$ where d is the exterior differential, d^* is the formal adjoint to d exterior codifferential and D is a conformal differential (see [16]).

Second, using basis operators, we construct the well-known *Hodge–de Rham Laplacian* $\Delta := d^*d + dd^*$ and the little-known *Tachibana operator* $\square := r(r+1)D^*D$.

We recall that the Hodge–de Rham operator is natural (that is, it commutes with isometric diffeomorphisms of Riemannian manifolds), elliptic, self-adjoint second order differential operator acting on differential forms on (M, g). If (M, g) is compact, spectrum of such operator is an infinite divergent sequence of real numbers, each eigenvalue being repeated according to its finite multiplicity (see [4, pp. 334–343]; [5, pp. 273–321]). Moreover, for a long time many geometers have been discussed the first eigenvalue of the Hodge–de Rham operator on a compact (M, g). They gave different estimates of its lower bounds on (M, g) (see, for example, [4,5,7,9,12,19,21,22] etc.).

Third, we note that the Tachibana operator $\square$ is natural (that is, it commutes with isometric diffeomorphisms of Riemannian manifolds), elliptic, self-adjoint second order differential operator acting on smooth differential forms on (M, g) too (see [4,18]). In addition, we will prove that $\square$ has positive eigenvalues, the eigenspaces of $\square$ are finite dimensional and eigenforms corresponding to distinct eigenvalues are orthogonal. Moreover, we will show that the first eigenvalue of the Tachibana operator on (M, g) with the positive curvature operator has its lover bound.

1.2. The present paper is based on our report at the international conference *Differential Geometry and Its Applications* (Czech Republic, August 19–23, 2013). At the same time the paper is a continuation of the research, which we started in [18] and [19].

2. Basic definitions and results

2.1. Let (M, g) be an n-dimensional compact connected Riemannian C^∞-manifold with some metric tensor g. For the metric g on M, we denote by ∇ the associated Levi-Civita connection, and by R and Ric respectively the Riemannian and Ricci curvature tensors of g. By $\Omega^r(M)$ we denote the vector space of C^∞-forms of degree $r = 0, 1, 2, \dots, n$ on (M, g). When the manifold (M, g) is oriented, we denote by ω_g the canonical n-form, called the *volume form* of (M, g).

The metric g induces pointwise inner products in fibers of various tensor bundles over (M, g). Then using the pointwise inner product in $\Omega^r(M)$ and the volume form ω_g we get implicitly the *Hodge star operator* $* : \Omega^r(M) \to \Omega^{n-r}(M)$ as the unique isomorphism mapping forms of degree r to forms of degree $n-r$ by the formula $\omega \wedge (*\theta) = g(\omega, \theta)\omega_g$ for all forms $\omega, \theta \in \Omega^r(M)$ (see [1, p. 33], [14, p. 203]).

By integrating the pointwise inner product $g(\omega, \theta)$ for any ω and θ in $\Omega^r(M)$ we get the *Hodge product* (see [1, p. 203])

$$\langle \omega, \theta \rangle = \int_M g(\omega, \theta)\omega_g = \int_M \omega \wedge *\theta = \int_M *\omega \wedge \theta. \tag{1}$$

2.2. Bourguignon proved (see [2]) the existence of a basis of the space of natural Riemannian (with respect to isometric diffeomorphisms) first-order differential operators on $\Omega^r(M)$ with values in the space of homogeneous tensors on (M, g) which consists of three operators, but he recognized only two of them. These operators are the well-known exterior differential $d : \Omega^r(M) \to \Omega^{r+1}(M)$ and the formal adjoint to d exterior codifferential $d^* : \Omega^{r+1}(M) \to \Omega^r(M)$ defined via the formula (see [14, p. 204])

$$\langle d\omega, \theta \rangle = \langle \omega, d^*\theta \rangle \tag{2}$$