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The dynamics of the Ricci flow on generalized Wallach spaces

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A B S T R A C T

We consider the asymptotic behavior of the normalized Ricci flow on generalized Wallach spaces that could be considered as a special planar dynamical system. All non-symmetric generalized Wallach spaces can be naturally parameterized by three positive numbers a_1, a_2, a_3. Our interest is to determine the type of singularity of all singular points of the normalized Ricci flow on all such spaces. Our main result gives a qualitative answer for almost all points (a_1, a_2, a_3) in the cube $(0, 1/2] \times (0, 1/2] \times (0, 1/2]$.

0. Introduction

The study of the normalized Ricci flow equation

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2\operatorname{Ric}_{\mathbf{g}} + 2\mathbf{g}(t)\frac{S_{\mathbf{g}}}{n} \tag{1}$$

for a 1-parameter family of Riemannian metrics $\mathbf{g}(t)$ in a Riemannian manifold M^n was originally used by R. Hamilton in [13] and since then it has attracted the interest of many mathematicians (cf. [7,25]). Recently,

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there is an increasing interest towards the study of the Ricci flow (normalized or not) on homogeneous spaces and under various perspectives ([1,5,6,12,14,17,22] and references therein).

The aim of the present work is to study the normalized Ricci flow for invariant Riemannian metrics on generalized Wallach spaces from a dynamical point of view to be explained later on. Generalized Wallach spaces are compact homogeneous spaces G/H whose isotropy representation decomposes into a direct sum $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ of three $\mathrm{Ad}(H)$-invariant irreducible modules satisfying $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ $(i \in \{1, 2, 3\})$ [21,19]. For a fixed bi-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of the Lie group G, any G-invariant Riemannian metric $\mathbf{g}$ on G/H is determined by an $\mathrm{Ad}(H)$-invariant inner product

$$(\cdot, \cdot) = x_1 \langle \cdot, \cdot \rangle\big|_{\mathfrak{p}_1} + x_2 \langle \cdot, \cdot \rangle\big|_{\mathfrak{p}_2} + x_3 \langle \cdot, \cdot \rangle\big|_{\mathfrak{p}_3}, \tag{2}$$

where x_1, x_2, x_3 are positive real numbers. By using expressions for the Ricci tensor and the scalar curvature in [21] the normalized Ricci flow equation (1) reduces to a system of ODEs of the form

$$\frac{dx_1}{dt} = f(x_1, x_2, x_3), \qquad \frac{dx_2}{dt} = g(x_1, x_2, x_3), \qquad \frac{dx_3}{dt} = h(x_1, x_2, x_3), \tag{3}$$

where $x_i = x_i(t) > 0$ $(i = 1, 2, 3)$, are parameters of the invariant metric (2) and

$$\begin{aligned}
f(x_1, x_2, x_3) &= -1 - \frac{A}{d_1} x_1 \left(\frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right) + 2x_1 \frac{S_{\mathbf{g}}}{n}, \\
g(x_1, x_2, x_3) &= -1 - \frac{A}{d_2} x_2 \left(\frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} \right) + 2x_2 \frac{S_{\mathbf{g}}}{n}, \\
h(x_1, x_2, x_3) &= -1 - \frac{A}{d_3} x_3 \left(\frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} \right) + 2x_3 \frac{S_{\mathbf{g}}}{n}, \\
S_{\mathbf{g}} &= \frac{1}{2} \left(\frac{d_1}{x_1} + \frac{d_2}{x_2} + \frac{d_3}{x_3} - A \left(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right) \right).
\end{aligned}$$

Here d_i, $i = 1, 2, 3$, are the dimensions of the corresponding irreducible modules $\mathfrak{p}_i$, $n = d_1 + d_2 + d_3$ and A is some special non-negative number (see Section 1). If $A \neq 0$, then by denoting $a_i := A/d_i > 0$, $i = 1, 2, 3$, the functions f, g, h can be expressed in a more convenient form (independent of A and d_i) as

$$\begin{aligned}
f(x_1, x_2, x_3) &= -1 - a_1 x_1 \left(\frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right) + x_1 B, \\
g(x_1, x_2, x_3) &= -1 - a_2 x_2 \left(\frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} \right) + x_2 B, \\
h(x_1, x_2, x_3) &= -1 - a_3 x_3 \left(\frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} \right) + x_3 B,
\end{aligned}$$

where

$$B := \left(\frac{1}{a_1 x_1} + \frac{1}{a_2 x_2} + \frac{1}{a_3 x_3} - \left(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right) \right) \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)^{-1}.$$

It is easy to check that the volume $V = x_1^{1/a_1} x_2^{1/a_2} x_3^{1/a_3}$ is a first integral of the system (3). Therefore, on the surface

$$V \equiv 1 \tag{4}$$