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A geodesic mapping and its field of symmetric linear endomorphisms

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ABSTRACT

It is well known that a geodesic mapping $f : (M, g) \to (\bar{M}, \bar{g})$ is defined through the tensor field A_f of type $(1, 1)$. In the present paper we give a geometric interpretation of eigenvalue function of the tensor field A_f and consider a global aspect of the geometry of the geodesic mapping $f : (M, g) \to (\bar{M}, \bar{g})$ with respect to the trace of A_f.

0. Introduction

Let (M, g) and $(\bar{M}, \bar{g})$ be two Riemannian manifolds of equal dimensions. The diffeomorphism $f : (M, g) \to (\bar{M}, \bar{g})$ is called a *geodesic mapping* [1, p. 127] if it maps any geodesic of (M, g) onto geodesic in $(\bar{M}, \bar{g})$. Moreover, if this mapping preserves the natural parameters of geodesics, then f is called *affine* [2, p. 225].

It is well known [1, p. 167], [2, p. 121] that a geodesic mapping $f : (M, g) \to (\bar{M}, \bar{g})$ is defined through the tensor field A_f of type $(1, 1)$. In the first section of the present paper we give a geometric interpretation of eigenvalues of the tensor field A_f. In the second section we consider a global aspect of the geometry of the geodesic mapping $f : (M, g) \to (\bar{M}, \bar{g})$ with respect to the trace of A_f.

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1. Geodesic mappings in terms of linear algebra

1.1. The necessary and sufficient condition for which an n-dimensional ($n \geqslant 2$) Riemannian manifold (M, g) to admit a geodesic mapping onto another n-dimensional Riemannian manifold $(\bar{M}, \bar{g})$ has the following form of the differential equation (see Eq. (8) in [2, pp. 120–122] and (7.7) in [1, pp. 167–168]):

$$(\nabla_Z a)(X, Y) = \theta(X)g(Y, Z) + \theta(Y)g(X, Z), \tag{1.1}$$

where ∇_Z denotes covariant derivative with respect to a smooth vector field Z on (M, g), a is a smooth covariant regular symmetric tensor field of order 2 on (M, g) (defined explicitly in the subsequent Remark) and θ is a gradient-like form for which $\theta := 1/2\,\mathrm{grad}(\mathrm{trace}_g\, a)$. In addition, we note that if $\theta = 0$ then the geodesic mapping $f : (M, g) \to (\bar{M}, \bar{g})$ is affine.

Remark. (See [2, pp. 120–121], [1, pp. 167–168].) Let $f : (M, g) \to (\bar{M}, \bar{g})$ be a geodesic mapping then in terms of local coordinate systems $x^1, \ldots, x^n$ of (M, g) and $\bar{x}^1, \ldots, \bar{x}^n$ of $(\bar{M}, \bar{g})$ we can suppose that f is represented by the equations $\bar{x}^1 = x^1, \ldots, \bar{x}^n = x^n$ for the corresponding points x and $\bar{x} = f(x)$. Also, let g_{ij} and $\bar{g}_{ij}$ be local components of the Riemannian metric tensors g and $\bar{g}$ for $i, j, k, l = 1, \ldots, n$. Denote by $\bar{g}^{ij}$ the elements of the inverse matrix $\|\bar{g}_{ij}\|^{-1}$, then the tensor a associated with the geodesic mapping $f : (M, g) \to (\bar{M}, \bar{g})$ is defined by $a_{ij} = e^{2\varphi}\bar{g}^{kl}g_{ik}g_{jl}$ where the function φ is defined by

$$\varphi = \frac{1}{2(n+1)}\big(\ln\det\|\bar{g}\| - \ln\det\|g\|\big).$$

Further we define the tensor field A_f, by setting $g(A_f X, Y) = a(X, Y)$ for arbitrary smooth tangent vector fields X and Y. In this case A_f is a self-adjoint section of $\mathrm{End}\, TM$, i.e. $g(A_f X, Y) = g(A_f Y, X)$. Moreover, with respect to the above definition Eq. (1.1) can be rewritten as

$$(\nabla_Y A_f)X = g(X, Y)\xi + \theta(X)Y, \tag{1.2}$$

for arbitrary smooth vector fields X, Y and ξ is given by $g(\xi, X) = \theta(X)$ for all X.

1.2. Let $\lambda_1(x), \ldots, \lambda_n(x)$ be the eigenvalues (some of which may coincide) of the symmetric endomorphism A_f of the tangent space T_xM at each $x \in M$. We can take an orthonormal basis $e_1, \ldots, e_n$ of T_xM such that $A_f e_i = \lambda_i(x)$, $1 \leqslant i \leqslant n$. Next, for an arbitrary $\lambda_i(x)$ we denote corresponding eigenspace by $D_{\lambda_i(x)} \subset T_xM$. It is well known that the geometric multiplicity of an eigenvalue $\lambda_i(x)$ is the dimension of the eigenspace $D_{\lambda_i(x)}$ associated to $\lambda_i(x)$, i.e. number of linearly independent eigenvectors with the eigenvalue $\lambda_i(x)$.

We shall denote by M_f a connected component of the open dense subset of M, which consists of points at which the number of all distinct eigenvalues $\lambda_1(x), \ldots, \lambda_r(x)$ of A_f is locally constant and the geometric multiplicities $n_1(x), \ldots, n_r(x)$ of the eigenvalues $\lambda_1(x), \ldots, \lambda_r(x)$ that are also locally constant. In this case, the eigenvalues of A_f form mutually distinct smooth eigenvalue functions as λ_α and, the assignment $x \in M_f \to D_{\lambda_\alpha(x)} \subset T_xM_f$ for all $x \in M_f$ defines a smooth eigenspace distribution D_{λ_α} of A_f. Then the following theorem is true.

Theorem 1. *Let $f : (M, g) \to (\bar{M}, \bar{g})$ be a geodesic mapping with its field of linear symmetric endomorphisms A_f and λ_α is an eigenvalue function of A_f, defined in a connected component of $M_f \subset M$. If the geometric multiplicity n_α of λ_α is at least two, then the eigenspace distribution D_{λ_α} is integrable and each maximal integral manifold of D_{λ_α} is an umbilical submanifold of (M, g) and λ_α is constant.*

Proof. Let $\lambda_\alpha = \lambda_\alpha(x)$ be an arbitrary eigenvalue functions of A_f, defined in a connected component of $M_f \subset M$. If the geometric multiplicity n_α of λ_α is at least two, then $\dim D_{\lambda_\alpha} = n_\alpha > 1$ for eigenspace