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Horospheres and hyperbolicity of Hadamard manifolds

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ABSTRACT

From geometrical study of horospheres we obtain, among asymptotically harmonic Hadamard manifolds, a rigidity theorem of the complex hyperbolic space $\mathbb{C}H^m$ with respect to volume entropy. We also characterize $\mathbb{C}H^m$ horospherically in terms of holomorphic curvature boundedness. Corresponding quaternionic analogues are obtained.

1. Introduction

Geometry of horospheres and their defining function, the Busemann function, is one of interesting geometrical subjects for nonpositively curved manifolds. Let (X, g) be an n-dimensional Hadamard manifold, i.e., a simply connected, complete Riemannian manifold of nonpositive curvature. In (X, g) a horosphere is defined as a level hypersurface $\mathcal{H} = \{x \in X \mid B_\theta(x) = \text{const.}\}$ of the Busemann function B_θ associated with a geodesic $\gamma = \gamma(t)$ which goes to an ideal point $\theta \in \partial X$ at infinity. The gradient field ∇B_θ and Hessian ∇dB_θ, respectively, stand for a unit normal field and the second fundamental form of the horosphere $\mathcal{H}$ whose hypersurface geometry can be described in terms of ∇dB_θ.

Horospheres of a typical Hadamard manifold have geometrically nice properties. In fact, a horosphere of the real hyperbolic space $\mathbb{R}H^n$ is flat, totally umbilic with constant principal curvature, and a horosphere of

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the complex hyperbolic space $\mathbb{C}H^m$ is characterized, from the results in [6], as one of Hopf real hypersurfaces with constant principal curvature among other tubular hypersurfaces. See also [7,8,14].

Taking an arbitrary real number t as level value of a Busemann function, we obtain, for a fixed $\theta \in \partial X$, a one parameter family of horospheres. In fact, for a given B_θ associated with a geodesic γ of $[\gamma] = \theta$ the horospheres $\{\mathcal{H}_{(\gamma(t),\theta)} = B_\theta^{-1}(-t) \mid t \in \mathbb{R}\}$, each of which passes through $\gamma(t)$ constitute a foliation of the ambient manifold X invariant by the geodesic flow. Geometric behavior of one parameter family of horospheres can be investigated by means of behavior of stable (or unstable) Jacobi tensor fields in time t along a geodesic γ.

An extremely important feature of stable (or unstable) Jacobi tensor fields is that along a geodesic γ tending to a $\theta \in \partial X$ they induce a one parameter family of shape operators $\mathcal{S}_t$, defined on the one parameter family of horospheres $\{\mathcal{H}_{(\gamma(t),\theta)} \mid t \in \mathbb{R}\}$. A one parameter family of shape operators $\{\mathcal{S}_t \mid t \in \mathbb{R}\}$ is a solution of the Riccati equation (for its precise definition see Section 3) so that $\{\mathcal{S}_t \mid t \in \mathbb{R}\}$ gives an appropriate tool for studying hypersurface geometry of horospheres. Here, we give an additional remark that the stable (or unstable) Jacobi tensor fields are also important in dynamical system of the geodesic flow on the unit sphere bundle of X. For this and behavior of the Anosov geodesic flow which is closely related to horospheres on a negatively curved closed manifolds, we refer to [5,13,18,22,35,37,49].

The purpose of this article is to present, from hypersurface geometry applied to one parameter families of horospheres, volume entropy rigidity theorems for the complex hyperbolic space $\mathbb{C}H^m$ and the quaternionic hyperbolic space $\mathbb{H}H^m$ (Theorems 1.5 and 1.7) and theorems which characterize $\mathbb{C}H^m$ and $\mathbb{H}H^m$ in terms of the value of second fundamental form $h(\cdot,\cdot)$ associated with structure vectors (Theorems 1.8 and 1.9).

In the volume entropy rigidity theorems an Hadamard manifold is assumed to be asymptotically harmonic. Here

Definition 1.1. (See [40].) An Hadamard manifold (X, g) is called *asymptotically harmonic* if $\Delta B_\theta(x)$ is a constant $-c$ for each $x \in X$ and $\theta \in \partial X$, where $\Delta = -\nabla^i\nabla_i$ is the Laplacian of the metric g.

The asymptotical harmonicity is equivalent to saying that the mean curvature of all horospheres in X is commonly constant $-c$. Here, by "mean" one means the sum of all principal curvatures. Furthermore (X, g) is asymptotically harmonic if and only if the positive function defined by $P(x, \theta) = \exp\{-cB_\theta(x)\}$ is harmonic on X for any $\theta \in \partial X$.

The motivation to our study is properly geometrical understanding of asymptotically harmonic Hadamard manifolds, since asymptotically harmonic manifolds appear in Fisher information geometry which plays a statistical role in the space $\mathcal{P}^+(\partial X, d\theta)$ of probability measures on the ideal boundary ∂X of an Hadamard manifold (X, g). As Theorem 1.3 in [31] illustrated, the constant $c > 0$ in Definition 1.1 appears as a homothety constant of the homothety map $\Phi : (X, g) \to (\mathcal{P}^+(\partial X, d\theta), G)$, where G is statistically defined metric over $\mathcal{P}^+(\partial X, d\theta)$, called Fisher information metric (see [1,23,31,34]).

Theorem 1.2. *(See [31].) Assume that an Hadamard manifold (X, g) admits a normalized Poisson kernel $P(x, \theta)$ (the fundamental solution to the Dirichlét problem at the ideal boundary; $\Delta u = 0$, $u_{|\partial X} = f$). Let Φ be a map from X to the space $\mathcal{P}^+(\partial X, d\theta)$, defined by $x \mapsto \mu(x) = P(x, \theta)\, d\theta$.*

Assume that the map Φ fulfills $\Phi^*G = \frac{c^2}{n} g$ ($c > 0$ is a constant) and further Φ is a harmonic map. Then, the Poisson kernel can be described as $P(x, \theta) = \exp\{-cB_\theta(x)\}$ in terms of B_θ (hence, $\Delta B_\theta = -c$, so (X, g) turns out to be asymptotically harmonic), and moreover (X, g) satisfies the axiom of visibility (see [20] for the notion of visibility).

The asymptotical harmonicity constant c appearing as the homothety constant in the above theorem has another geometrical meaning. The constant c coincides with the volume entropy $\rho(X)$ of X, as indicated in [33].