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Walker manifolds and Killing magnetic curves

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ABSTRACT

On a Walker manifold $\mathcal{M}_f^3$, we first characterize the Killing vector fields, aiming to obtain the corresponding Killing magnetic curves. When the manifold is endowed with a unitary spacelike vector field ξ, we prove that after a reparameterization, any lightlike curve normal to ξ is a lightlike geodesic. We also show that on $\mathcal{M}_f^3$, equipped with a Killing vector field V, any arc length parameterized spacelike or timelike curve, normal to V, is a magnetic trajectory associated to V. We characterize the normal magnetic curves corresponding to some Killing vector fields on $\mathcal{M}_f^3$, obtaining their explicit expressions for certain functions f.

1. Introduction

Walker manifolds, described in the monograph [4], published in 2009, have various applications in mathematics and theoretical physics, as one can see in the References of the above-mentioned work, as well as in [11] and [14].

We work in the context of a pseudo-Riemannian manifold (i.e. a manifold equipped with a non-degenerate metric tensor of arbitrary signature). More precisely, we develop our study on a Walker space, defined as a pseudo-Riemannian manifold with a lightlike distribution (see [9]), which is parallel with respect to the Levi-Civita connection. Pseudo-Riemannian manifolds (in particular Lorentzian) are important in physics, due to their applications to general relativity.

A notion appeared for the first time in physics is that of magnetic curve, or magnetic trajectory, describing the motion of a charged particle, under the influence of a magnetic field. In mathematics, magnetic curves

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were studied first on Riemannian surfaces (see e.g. [15,3]), then in 3-dimensional context, on $\mathbb{E}^3$ [7], $\mathbb{E}^3_1$ [8], $\mathbb{S}^3$ [2], $\mathbb{S}^2 \times \mathbb{R}$ [12], etc. Recently, a series of papers (see [1,5], and the references therein) are devoted to magnetic curves on arbitrary dimensional Kähler and contact manifolds.

Concerning the geometric meaning of magnetic curves in the framework of pseudo-Riemannian geometry, we recall that some geometric aspects were obtained in [8] and some others will be presented in a forthcoming paper.

Our goal now is to investigate magnetic curves on Walker manifolds, in order to classify the corresponding Killing magnetic curves. To this purpose we focus on Walker manifolds of dimension three, denoted by $\mathcal{M}^3_f$ as in [4], for whose study we are motivated by [6]. First, we classify the Killing vector fields on these spaces. For the existence of such vector fields, we provide some restrictions of the Walker metric. As a consequence, we obtain that the vector field ∂_x is Killing on $\mathcal{M}^3_f$, if and only if this Walker manifold is strict, according to the terminology of the book [4].

Geodesics can be viewed as a particular type of magnetic trajectories, obtained in the case when the magnetic field vanishes, namely when the charged particle moves under the influence of gravity, only. On pseudo-Riemannian manifolds, lightlike geodesics are of particular interest (see [10] and the references therein). If a Walker manifold $\mathcal{M}^3_f$ carries a unitary spacelike vector field ξ, which is normal to a lightlike curve γ (i.e. γ is an integral curve of $\xi^\perp$), then after a reparameterization, γ is a geodesic. Moreover, when $\mathcal{M}^3_f$ is endowed with a Killing vector field V, we obtain the magnetic curves corresponding to V, by considering the integral spacelike or timelike curves of $V^\perp$. Finally, we characterize the normal magnetic trajectories associated to the Killing vector field ∂_x on $\mathcal{M}^3_f$, and obtain some examples of Killing magnetic curves on such manifolds.

Notations. On a pseudo-Riemannian manifold, an arbitrary parameterized curve will be denoted by $\gamma(t)$, and its speed by $\gamma'(t)$. Since a lightlike curve cannot be parameterized by its arc length, from now on, $\gamma(s)$ will denote an arc length parameterized curve, which is called spacelike or timelike, just like its speed vector, denoted by $\dot{\gamma}(s)$.

Definition 1.1. A *magnetic field* on a pseudo-Riemannian manifold (M, g) is defined as a closed 2-form F on M, related to a skew-symmetric $(1,1)$-tensor field Φ, called the *Lorentz force* of F, by:

$$g\big(\Phi(X), Y\big) = F(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{1}$$

Note that the Lorentz force is divergence free, i.e. $\operatorname{div}\Phi = 0$.

A *magnetic potential* on a manifold M is a 1-form $A \in \Lambda^1(M)$, such that the magnetic field is $F = dA \in \Lambda^2(M)$.

A *magnetic trajectory* of F (or a *magnetic curve* associated to F) is a smooth curve γ on M, satisfying the *Lorentz equation* (called also *Newton*, or *Landau–Hall equation*):

$$\nabla_{\gamma'}\gamma' = q\Phi\big(\gamma'\big), \tag{2}$$

where ∇ is the Levi-Civita connection of g, and q is the charge of the particle.

Obviously, Eq. (2) generalizes the equation of the geodesics of M. A magnetic trajectory is *a flowline of the dynamical system*, associated to the magnetic field, while a geodesic can be viewed as a trajectory of a particle moving on the manifold in the absence of a magnetic field.

Due to the skew-symmetry of the Lorentz force, the magnetic trajectories have constant speed $v(t) = \|\gamma'\| = v_0$ (see e.g. [5]). When the magnetic curve $\gamma(t)$ is arc length parameterized ($v_0 = 1$), it is called a *normal magnetic curve*.