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A certain two-parameter family of helices of order 6 in Euclidean sphere [☆]

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ABSTRACT

In this paper, using a fundamental fact on circles of a complex projective space $\mathbb{C}P^n(c)$ and a well-known minimal embedding of a complex projective plane $\mathbb{C}P^2(c)$ into a 7-dimensional sphere, we find a nice family of helices of order 6 on Euclidean sphere $S^n(c)$, $n \geqslant 6$.

1. Introduction

We take small circles of positive curvature k on an n $(\geqslant 2)$-dimensional sphere $S^n(c)$ of constant sectional curvature c (> 0). It is well known that such small circles are closed curves of length $2\pi/\sqrt{k^2+c}$. Needless to say, this length is shorter than $2\pi/\sqrt{c}$ which is the length of great circles. Since all circles can be regarded as helices of order 2, it is natural to pose the following

Question. *When $n \geqslant 3$, for every positive constant ρ, does there exist a closed helix whose length is ρ on $S^n(c)$?*

The purpose of this paper is to give the following partial affirmative answer to this question.

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Answer. *When $n \geqslant 6$, for every positive constant ρ, there exists a closed helix of order 6 whose length is ρ on $S^n(c)$.*

In this paper, such closed helices cannot be obtained by variational methods. Using the theory of Riemannian submanifolds in Riemannian symmetric spaces of rank one, we obtain desired closed helices of any length on an n ($\geqslant 6$)-dimensional Euclidean sphere $S^n(c)$. In the next section, we explain this idea in detail in order to obtain our Answer.

2. Background and ideas in our research

We first explain the relation between the notion of homogeneous curves and the notion of helices in Riemannian Geometry. A real curve $\gamma = \gamma(s)$ parameterized by its arclength s on a Riemannian manifold M is called a *homogeneous curve* if γ is an orbit of some one-parameter subgroup of the isometry group $\mathrm{I}(M)$ of M. Then all curvature functions of the curve γ are automatically constant functions along γ, namely γ is a helix on M (for the definition of helices, see Section 3). Moreover, every homogeneous curve γ is a simple curve because the curve γ is an integral curve of some Killing vector field on M. It is well known that in an n-dimensional real space form $M^n(c)$ of constant sectional curvature c (namely, $M^n(c)$ is either $S^n(c)$, an n-dimensional Euclidean space $\mathbb{R}^n$ or an n-dimensional real hyperbolic space $H^n(c)$ according as c is either positive, zero or negative), a curve $\gamma = \gamma(s)$ is a homogeneous curve if and only it is a helix. This means that every helix in an n-dimensional real space form $M^n(c)$ is a simple curve. However, in general there exist non-simple helices on a Riemannian manifold M which is not congruent to $M^n(c)$. For example, in a complex projective plane $\mathbb{C}P^2(c)$ of constant holomorphic sectional curvature c there exists a non-simple closed helix of proper order 3 with curvatures $k_1 = \sqrt{3}/\sqrt{c}$, $k_2 = \sqrt{6}/\sqrt{c}$ whose length is $2\pi\sqrt{c}$. Note that it has 6 self-intersection points (see Theorem 1 in [5]).

We next explain some fundamental properties of circles in a Riemannian symmetric space M. It is well known that every geodesic on a Riemannian symmetric space M is a homogeneous curve. We here take a Riemannian symmetric space M of *rank one*, that is M is one of $S^n(c)$, an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(> 0)$, an n-dimensional quaternionic projective space $\mathbb{H}P^n(c)$ of constant quaternionic sectional curvature $c(> 0)$, the Cayley projective plane $\mathbb{C}ayP^2(c)$ of maximal sectional curvature $c(> 0)$ or these noncompact duals $\mathbb{R}H^n(c)$, $\mathbb{C}H^n(c)$, $\mathbb{H}H^n(c)$, $\mathbb{C}ayH^2(c)$ with $c < 0$. Then we know that all circles including geodesics in a Riemannian symmetric space M of rank one are homogeneous curves (see [6]). Motivated by this fact, Mashimo and Tojo [7] establish the following:

Theorem A. *Let M be a Riemannian homogeneous space. Then M is either a Euclidean space or a Riemannian symmetric space of rank one if and only if there exists a positive constant k satisfying that every circle of positive curvature k on M is a homogeneous curve.*

Moreover, we can see the following (cf. [1,3,4,7]):

Theorem B.

(1) *For every positive constant ρ there exits a closed circle γ with length ρ in a Riemannian symmetric space M of rank one which is not congruent to a real space form.*
(2) *Every circle in a compact (resp. noncompact) Riemannian symmetric space M of rank one, which is not congruent to a real space form, lies on a totally geodesic submanifold $\mathbb{C}P^2(c)$ (resp. $\mathbb{C}H^2(c)$) of $\mathbb{C}P^n(c), \mathbb{H}P^n(c)$ and $\mathbb{C}ayP^2(c)$ (resp. $\mathbb{C}H^n(c), \mathbb{H}H^n(c)$ and $\mathbb{C}ayH^2(c)$) as a circle, where $n \geqslant 2$.*