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Invariants of Riemannian curves in dimensions 2 and 3

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ABSTRACT

In this article, a complete and geometrical description of minimal sets of differential invariants in 2- and 3-dimensional Riemannian manifolds is given in terms of the dimension of the isometry group. These invariants provide a solution to the (local) problem of congruence of curves so that a final generalization of Frenet's theory in low dimensional manifolds is reached.

1. Introduction

Two curves $\sigma, \bar{\sigma} : (a,b) \to M$ taking values in a Riemannian manifold (M,g) are said to be congruent if there exists an isometry $\varphi \in \mathcal{I}(M,g)$ such that $\bar{\sigma} = \varphi \circ \sigma$. The solution of the problem of congruence in the Euclidean space $\mathbb{R}^3$ is among the most celebrated results in Differential Geometry. This characterization is formulated by means of the arc-length, the curvature, and the torsion functions defined by the theory developed by Frenet and Serret in the XIX century. Since then, this topic has acquired much attention from geometers either in particular manifolds (manifolds of constant curvature, symmetric manifolds, etc.) or more general settings. The main tools used for the congruence of curves deal with moving frames, groups of isometries, the geometry of homogeneous manifolds, differential invariants or their combinations. The reader can find interesting results and references on the topic in, for example, [5,6,12,13].

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The existence of global isometries in a manifold depends on several factors, the completeness and its topology among them. This indicates that the problem of congruence may have a richer behaviour if studied from a local point of view. In the reference [2], the authors provide full answer to the problem of congruent curves in local terms. In particular, one has that the Frenet curvatures do not suffice to give the complete characterization of congruence and more invariants, built from the Riemann curvature tensor and its covariant derivatives, are needed. These invariants are not functionally independent generically. In fact, a deeper study shows that a minimal set of invariants can be obtained with just $m = \dim M$ elements. The rest of invariants are function of these and their total derivatives. Unfortunately practise teaches us that the precise determination of the invariants belonging to these minimal sets must be obtained case by case.

The purpose of the present article is the determination of minimal set of invariants for the (local) congruence problem in 2- and 3-dimensional manifolds together with their geometric description. This represents a final generalization of Frenet's theory in low dimensional Riemannian manifolds. The structure of this article is organized in three sections. The first one gives an account of the main results of [2] which are needed in the following. The second section studies a bound on the number of covariant derivatives of the curvature tensor to be added to the Frenet curvature in the case of surfaces. Finally, the last section provides the study in 3-manifolds and contains the main results of the article.

2. The problem of congruence on Riemannian manifolds

The reader can found the proofs of the results contained in this section in the reference [2], to which we also refer for further literature and definitions.

A curve $\sigma : (a,b) \to M$ taking values in a Riemannian manifold (M,g), $m = \dim M$, is said to be a Frenet curve if the tangent vectors $\{\nabla^i_{T^\sigma} T^\sigma\}$, $i = 0,\dots,m-1$, are linearly independent for any $t \in (a,b)$, where $T^\sigma(t) = \sigma'(t)$ and ∇ stands for the Levi-Civita connection. This condition is generic in the sense that the set of Frenet curves is a dense open subset in $C^\infty(\mathbb{R}, M)$ with respect to the strong topology. The definition of the Frenet frame is a consequence of the following result:

Proposition 2.1. *If (M,g) is an oriented connected Riemannian manifold of dimension m and $\sigma : (a,b) \to M$ is a Frenet curve, then there exist unique vector fields $X^\sigma_1,\dots,X^\sigma_m$ defined along σ and smooth functions $\kappa^\sigma_0,\dots,\kappa^\sigma_{m-1} : (a,b) \to \mathbb{R}$ with $\kappa^\sigma_j > 0$, $0 \leqslant j \leqslant m-2$, such that,*

- (i) $(X^\sigma_1(t),\dots,X^\sigma_m(t))$ *is a positively oriented orthonormal linear frame, $\forall t \in (a,b)$.*
- (ii) *The systems $(X^\sigma_1(t),\dots,X^\sigma_i(t))$, $(T^\sigma_t, (\nabla_{T^\sigma}T^\sigma)_t,\dots,(\nabla^{i-1}_{T^\sigma}T^\sigma)_t)$ span the same vector subspace and they are equally oriented for every $1 \leqslant i \leqslant m-1$ and every $t \in (a,b)$.*
- (iii) *The following formulas hold*:
  - (a) $T^\sigma = \kappa^\sigma_0 X_1$,
  - (b) $\nabla_{X^\sigma_1} X^\sigma_1 = \kappa^\sigma_1 X^\sigma_2$,
  - (c) $\nabla_{X^\sigma_1} X^\sigma_i = -\kappa^\sigma_{i-1} X^\sigma_{i-1} + \kappa^\sigma_i X^\sigma_{i+1}$, $2 \leqslant i \leqslant m-1$,
  - (d) $\nabla_{X^\sigma_1} X^\sigma_m = -\kappa^\sigma_{m-1} X^\sigma_{m-1}$.

The functions $\kappa^\sigma_0,\dots,\kappa^\sigma_{m-1}$ are called Frenet or principal curvatures. They provide properties of the extrinsic geometry of σ in (M,g) and encode enough information to reconstruct, at least locally, a Frenet curve. More precisely, we have

Theorem 2.2. *Let (M,g) be an m-dimensional oriented Riemannian manifold and let $(v_1,\dots,v_m)$ be a positively oriented orthonormal basis for $T_{x_0}M$. Given functions $\kappa_j \in C^\infty(t_0-\delta, t_0+\delta)$, $0 \leqslant j \leqslant m-1$,*