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On the geometry of Weil bundles

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ABSTRACT

We demonstrate how the covariant approach to Weil bundles can be used for deducing some general geometric results. Special attention is paid to the Weilian prolongations of tangent valued forms and connections.

In [7], A. Weil introduced an infinitely near point on a smooth manifold M as an algebra homomorphism of the algebra $C^\infty(M,\mathbb{R})$ of smooth functions on M into a local algebra A. Nowadays A is called Weil algebra and the space of the corresponding near point $T^A M$ on M is said to be a Weil bundle. About 1985, it was deduced that the product preserving bundle functors of the category of all smooth manifolds $\mathcal{M}f$ are just the Weil functors, see [6]. This result clarified that the Weil bundles should be a good instrument for differential geometry. Further, in [5] the authors deduced that the fiber product preserving bundle functors on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps can also be described in terms of Weil algebras. A survey of some results in both these directions can be found in [3].

In the present paper we discuss the classical Weil functors and deduce some of their general geometric properties. Section 1 is introductory. In Section 2 we study the covariant form of the natural transformations of Weil functors. In Section 3 we discuss the prolongation of tangent valued forms. In particular, we justify an interesting infinitesimal-like algorithm. In the last section we study the Weilian prolongations of connections.

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We consider the classical smooth manifolds and maps, i.e. all manifolds are finite dimensional and smooth means C^∞-differentiable. All manifolds are assumed to be Hausdorff and separable. Unless otherwise specified, we used the terminology and notation from the book [6].

1. The covariant approach

By definition, a Weil algebra A is a finite dimensional, commutative, associative and unital algebra of the form $\mathbb{R} \times N$, where $\mathbb{R}$ denotes the real multiples of the unit 1_A of A and N is the ideal of all nilpotent elements. According to A. Weil, the bundle of infinitely near point of type A

$$T^A M = \operatorname{Hom}\bigl(C^\infty M, A\bigr) \tag{1}$$

is the set of all algebra homomorphisms. Every map $f : M \to \bar{M}$ induces $f^* : C^\infty \bar{M} \to C^\infty M$. For $\varphi \in \operatorname{Hom}(C^\infty M, A)$, we put

$$T^A f(\varphi) = \varphi \circ f^* \in \operatorname{Hom} C^\infty(\bar{M}, A). \tag{2}$$

This defines a bundle functor $T^A : \mathcal{M}f \to \mathcal{FM}$ called Weil functor. By (2), this approach is said to be contravariant. One verifies easily that T^A preserves products. If B is another Weil algebra and $\mu \in \operatorname{Hom}(A, B)$ is an algebra homomorphism, the rule

$$\mu_M(\varphi) = \mu \circ \varphi \in \operatorname{Hom}\bigl(C^\infty M, B\bigr) \tag{3}$$

defines a natural transformation $\mu_M : T^A M \to T^B M$.

The simplest example of a Weil algebra is

$$\mathbb{D}^r_k = \mathbb{R}[x_1, \dots, x_k]/\langle x_1, \dots, x_k\rangle^{r+1}, \tag{4}$$

where $\mathbb{R}[x_1, \dots, x_k]$ is the algebra of all polynomials in k undetermined. In particular, $\mathbb{D}^1_1 = \mathbb{D}$ is the algebra of dual numbers.

About 1986, the following fundamental result was deduced, see [6] for details. Let $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the multiplication of reals.

Theorem 1. *Let F be a product preserving bundle functor on $\mathcal{M}f$. Then $F\mathbb{R}$ is a Weil algebra with respect to the multiplication Fm and F coincides with the Weil functor $T^{F\mathbb{R}}$. The natural transformations $t : T^A \to T^B$ are in bijection with the algebra homomorphisms $t_\mathbb{R} : A \to B$.*

One finds easily that $T^{\mathbb{D}^r_k}$ coincides with the functor T^r_k of (k, r)-velocities by C. Ehresmann, [3]. In particular, $T^1_1 = T$ is the classical tangent functor.

Since $A = \mathbb{R} \times N$ is finite dimensional, there exists an integer r such that $N^{r+1} = 0$. The smallest r with this property is called the order of A. On the other hand, the dimension wA of the vector space N/N^2 is said to be the width of A. A Weil algebra of width k and order r will be called Weil (k, r)-algebra. In [3], we deduced

Lemma. *Every Weil (k, r)-algebra A is a factor algebra of $\mathbb{D}^r_k$. If π, $\varrho : \mathbb{D}^r_k \to A$ are two surjective algebra homomorphisms, then there is an algebra isomorphism $\sigma : \mathbb{D}^r_k \to \mathbb{D}^r_k$ satisfying $\pi = \varrho \circ \sigma$.*

In [3] we developed systematically the following covariant approach to Weil bundles. Since π is determined up to an isomorphism $\mathbb{D}^r_k \to \mathbb{D}^r_k$, the following definition is independent of the choice of π.