Differential Geometry and its Applications 35 (2014) 255–265

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Pushing down the Rumin complex to conformally symplectic quotients ☆

Andreas Čap *, Tomáš Salač

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

ARTICLE INFO

Article history:
Received 11 December 2013
Received in revised form 16 May 2014
Available online 28 May 2014
Communicated by J. Slovák

MSC:
primary 58J10
secondary 35N10, 53D05, 53D10, 58J60

ABSTRACT

Given a contact manifold $M_{\#}$ together with a transversal infinitesimal automorphism ξ, we show that any local leaf space M for the foliation determined by ξ naturally carries a conformally symplectic (cs-) structure. Then we show that the Rumin complex on $M_{\#}$ descends to a complex of differential operators on M, whose cohomology can be computed. Applying this construction locally, one obtains a complex intrinsically associated with any manifold endowed with a cs-structure, which recovers the generalization of the so-called Rumin–Seshadri complex to the conformally symplectic setting. The cohomology of this more general complex can be computed using the push-down construction.

1. Introduction

This article is motivated by the results [8] of M. Eastwood and H. Goldschmidt on integral geometry and the subsequent work [9] of M. Eastwood and J. Slovák on conformally Fedosov structures. The main tool used in [8] is a family of complexes of differential operators on complex projective space $\mathbb{C}P^n$. The results on integral geometry are deduced from vanishing of some cohomology groups of these complexes. The form and length of these complexes is rather intriguing and the article [9] takes steps towards an explanation. The main notion introduced there is the one of a conformally Fedosov structure, which combines a conformally symplectic structure and a projective structure, which satisfy a suitable compatibility condition. Given these data, the authors construct a tractor bundle endowed with a (linear) tractor connection which is naturally associated with the conformally Fedosov structure. This should open the possibility to construct sequences and complexes of differential operators following the ideas of the Bernstein–Gelfand–Gelfand (BGG) machinery as introduced in [6] and [4] in the setting of parabolic geometries.

☆ Both authors gratefully acknowledge support by project P23244-N13 of the "Fonds zur Förderung der wissenschaftlichen Forschung" (FWF).
We also want to thank Michael Eastwood, who (after the first version of this article was submitted) pointed out his article [7] to us.

* Corresponding author.
E-mail addresses: Andreas.Cap@univie.ac.at (A. Čap), Tomas.Salac@univie.ac.at (T. Salač).

http://dx.doi.org/10.1016/j.difgeo.2014.05.004
0926-2245/

The tractor bundle associated with a conformally Fedosov structure looks similar to the standard tractor bundle associated with a contact projective structure (see [10]). This is an instance of a so-called *parabolic contact structure*, the best known example of which are (hypersurface type) CR structures. It is known that the homogeneous models of parabolic contact structure are, via forming quotients by transversal infinitesimal automorphisms, related to special symplectic connections, see [3] and Sections 5.2.18 and 5.2.19 of [5] for an exposition in the language of parabolic geometries.

The starting point for our considerations is the hope to obtain complexes like the ones constructed in [8] from BGG sequences associated with parabolic contact structures via similar quotient constructions. In this article, we show that this can indeed be done in the special case of the BGG sequence associated with the trivial representation. It is shown in [2] that in this case one obtains the Rumin complex (see [11]), which can be naturally constructed for any contact structure, see [2] for a simple direct construction. From either construction it follows that the Rumin complex is a fine resolution of the constant sheaf $\mathbb{R}$, so in particular it computes the de-Rham cohomology of a contact manifold. Following this, our article also works in the setting of general contact manifolds and does not use parabolic geometry techniques.

We first prove that the quotient of a contact structure by a transversal infinitesimal automorphism naturally inherits a symplectic structure, and thus in particular a conformally symplectic structure (or cs-structure). In contrast to the traditional approach to defining such a structure via a specific two-form, we just view it as an appropriate line subbundle in the bundle of two-forms, which simplifies matters in several respects.

Next, we show that the Rumin complex can be pushed down to a complex of differential operators on the quotient space, which coincides with the complex on a symplectic manifold constructed in [13,14] and in [2], where it is called *Rumin–Seshadri* complex, see also [12]. The push-down construction easily leads to a long exact sequence relating the cohomology of this complex to de-Rham cohomology. In particular, one can immediately read off (in this very simple special case) the cohomological information needed in the applications in [8].

To complete the picture, we prove that the push down construction can be used to construct a version of this complex and an analog of the long exact sequence on any smooth manifold endowed with a cs-structure. We prove that any such manifold can be locally represented as the quotient of a contact structure by a transversal infinitesimal automorphism. Moreover, any local isomorphism of cs-structures can be lifted to a contactomorphism “upstairs”. Naturality of the Rumin complex under contactomorphisms then implies that one can use the local contactifications to obtain a complex and a long exact sequence on the whole cs-manifold and that they are intrinsic to the cs-structure.

The extension of the construction of the Rumin–Seshadri complex from symplectic manifolds to cs-manifolds is not a new result in its own right, a direct construction is available in [7]. The main advantage of our approach is not the result itself, but the potential for generalizations.

2. A quotient of the Rumin complex

We start by discussing local quotients of contact manifolds by transverse infinitesimal automorphisms.

2.1. Contact manifolds and differential forms

By a contact manifold, we will always mean a manifold $M_\#$ of odd dimension $2n+1$ endowed with a maximally non-integrable distribution $H \subset TM$ of rank $2n$. While this implies that locally H can be written as the kernel of a one-form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing, we do not initially assume that such a *contact form* exists globally or that there locally is a preferred choice.

The best way to conceptually formulate the condition of maximal non-integrability is via the *Levi-bracket*. Defining $Q := TM_\#/H$, which clearly is a line bundle naturally associated with any corank one subbundle in